ON THE THEORY OF FUZZY RELATIONS

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Abstract

In this paper, we seek the reality to the theory of fuzzy relations. For that we need to discuss about some linguistic relation which across some definitions and establish some operations, properties and composition of fuzzy relations.

Keywords: Fuzzy logic, Fuzzy sets, Fuzzy relations, Composition of fuzzy relations, Cartesian product.

1. Introduction

One of the most fundamental notions in pure and applied sciences is the concept of a relation. Science has been described as the discovery of relations between objects, states and events. Fuzzy relations generalize the concept of relations in the same manner as fuzzy sets generalize the fundamental idea of sets.

A relation is a mathematical description of a situation where the certain elements of sets are related to one another in some way. Fuzzy relations are significant concepts in the fuzzy theory to play an important role in fuzzy modeling, fuzzy diagnosis and fuzzy control. They also have applications in fields such as psychology, medicine, economics and sociology.

Fuzzy relation was introduced by Zadeh (1971) as a generalization of bags classical relations and fuzzy set. In this paper we discuss about some linguistic relation and establish some basic operations on fuzzy relations. It represents the strength of association between elements of two sets.

2. Definitions of fuzzy relations

Consider the Cartesian product $A \times B = \{(x,y): x \in A , y \in B\}$. Where A and B are subsets of the universal sets U_1 and U_2correspondingly.A fuzzy relation on $A \times B$ denoted by R or $R(x,y)$ is defined as the set

(2.1) $R = \{ (x,y), \mu_R(x,y) : (x,y) \in A \times B , \mu_R(x,y) \in [0,1]\}$

Where $\mu_R(x,y)$ is a function in two variables called membership function. It gives the degree of membership of the ordered pair (x,y) in R associating with each pair (x,y) in $A \times B$ a real number in the interval [0,1]. The degree of membership indicates the degree to which x is in the relation with y. Formally the fuzzy relation R is a classical trinary relation, it is a set of ordered triples.

The above definition is a generalization of definition of fuzzy set from two dimensional space $(x, \mu_A(x))$ to three dimensional space $(x, y, \mu_A(x,y))$. Here we also identify a relation with its membership function.

The Cartesian product $A \times B$ in fuzzy relation is a relation by itself between x and y in the sense of classical set theory. It can be considered as a zero fuzzy relation.

(2.2) $0 = \{\big((x,y), \mu_0(x,y)\big) : (x,y) \in A \times B , \mu_0(x,y) = 0\}$

If we set $\mu_R(x,y) = 1$ in the definition of fuzzy relation ,we get a classical relation .

The identity relation I is defined for all $(x,y) \in A \times B$ by its membership function as follows

(2.3) $I = \mu_I(x,y) = \begin{cases} 1 , & for\ x = y \\ 0 , & for\ x \neq y \end{cases}$

The fuzzy relation R is a subset of a cartesian product in three dimensional space, $i.e$

(2.4) $\{((x,y),\mu_R(x,y)) \in (A \times B) \times [0,1] \subset U_1 \times U_2 \times [0,1]\}$

The fuzzy relation in comparison to the classical relation possesses stronger expressive power while relating x and y due to the membership function $\mu_R(x,y)$ which assigns specific grades to each pair (x,y) .

Common linguistic relations that can be described by appropriate fuzzy relations are : x is much greater than y , x is close to y , x is relevant to y , x and y are almost equal,x and y are very far etc .

Example 2.1

The fuzzy relation

$$R = \{((x_1,y_1),0),((x_1,y_2),0.1),((x_1,y_3),0.2), ((x_2,y_1),0.7),((x_2,y_2),0.2),((x_2,y_3),0.3), ((x_3,y_1),1),((x_3,y_2),0.6),((x_3,y_3),0.2)\}$$

Can be given also by the table

$R =$

x \ y	y_1	y_2	y_3
x_1	0	0.1	0.2
x_2	0.7	0.2	0.3
x_3	1	0.6	0.2

Where the number in the cells located at the intersection of rows and columns are the values of the membership function:-

$$\mu_R(x_1,y_1) = 0 \ , \ \mu_R(x_1,y_2) = 0.1 \ , \ \mu_R(x_1,y_3) = 0.2$$

$$\mu_R(x_2,y_1) = 0.7 \ , \ \mu_R(x_2,y_2) = 0.2 \ , \ \mu_R(x_2,y_3) = 0.3$$

$$\mu_R(x_3,y_1) = 1 \ , \ \mu_R(x_3,y_2) = 0.6 \ , \ \mu_R(x_3,y_3) = 0.2$$

Example 2.2

Assume that two universes $A = \{3,4,5\}$ and $B = \{3,4,5,6,7\}$ such that

$$\mu_R(x,y) = \begin{cases} (y-x)/(y+x+2) & if\ y > x \\ 0 & if\ y \le x \end{cases}$$

Which can be expressed by the table

$R =$

A \ B	3	4	5	6	7
3	0	o.11	0.2	0.27	0.33
4	0	0	0.09	0.17	0.23
5	0	0	0	0.08	0.14

3. Basic Operations on Fuzzy relations

Let R_1 and R_2be two fuzzy fuzzy relations on $A \times B$ such that

$$R_1 = \{((x,y),\mu_{R_1}(x,y)\} \ , \ (x,y) \in A \times B \ .$$

$$R_2 = \{((x,y),\mu_{R_2}(x,y)\} \ , \ (x,y) \in A \times B \ .$$

We use the membership functions $\mu_{R_1}(x,y)$ and $\mu_{R_2}(x,y)$ in order to introduce operations with R_1 and R_2 similarly to operations with fuzzy sets.

3.1 Equality

$R_1 = R_2$ iff for every pair $(x,y) \in A \times B$, we have $\mu_{R_1}(x,y) = \mu_{R_2}(x,y)$.

3.2 Inclusion

For every pair $(x,y) \in A \times B$, then $\mu_{R_1}(x,y) \le \mu_{R_2}(x,y)$, the relation R_1 is included in R_2 or $R_2$is larger than R_1 ,denoted by $R_1 \subseteq R_2$. If $R_1 \subseteq R_2$ and in addition if for at least one pair (x,y) ,$\mu_{R_1}(x,y) < \mu_{R_2}(x,y)$,then we have the proper inclusion $R_1 \subset R_2$.

Example 3.1 For the relations

$R_1 =$

Y X	y_1	y_2	y_3
x_1	0	0.2	0.6
x_2	0.4	1.0	0.8

$R_2 =$

Y X	y_1	y_2	y_3
x_1	0.1	0.2	0.7
x_2	0.5	1.0	0.9

Then we have

$$\mu_{R_1}(x_1, y_1) = 0 < \mu_{R_2}(x_1, y_1) = 0.1$$

$$\mu_{R_1}(x_2, y_1) = 0.4 < \mu_{R_2}(x_2, y_1) = 0.5$$

$$\mu_{R_1}(x_1, y_2) = 0.2 = \mu_{R_2}(x_1, y_2) = 0.2$$

$$\mu_{R_1}(x_2, y_2) = 1.0 = \mu_{R_2}(x_2, y_2) = 1.0$$

$$\mu_{R_1}(x_1, y_3) = 0.6 < \mu_{R_2}(x_1, y_3) = 0.7$$

$$\mu_{R_1}(x_2, y_3) = 0.8 < \mu_{R_2}(x_2, y_3) = 0.9$$

Hence R_1 is included in R_2 $i.e. R_1 \subset R_2$.

3.3 Complementation

The complement of a relation R, denoted by $\bar{R}$, is defined by

$$\mu_{\bar{R}}(x, y) = 1 - \mu_R(x, y)$$

Which must be valid for any pair

$(x, y) \in A \times B$.

Example 3.2

The complements of the relations R_1 *and* R_2 given in example (3.1), according to the definition of complement of a relation, we have

$\overline{R_1} =$

Y X	y_1	y_2	y_3
x_1	1.0	0.8	0.4
x_2	0.6	0	0.2

$\overline{R_2} =$

Y X	y_1	y_2	y_3
x_1	0.9	0.8	0.3
x_2	0.5	0	0.1

3.4 Union

The union of R_1 *and* R_2 denoted by $R_1 \cup R_2$ is defined by

$\mu_{R_1 \cup R_2}(x, y) = max\{ \mu_{R_1}(x, y) , \mu_{R_2}(x, y)\}$, $(x, y) \in A \times B$.

3.5 Intersection

The intersection of R_1 *and* R_2 denoted by $R_1 \cap R_2$ is defined by

$\mu_{R_1 \cap R_2}(x, y) = min\{ \mu_{R_1}(x, y) , \mu_{R_2}(x, y)\}$, $(x, y) \in A \times B$.

Example 3.3

Consider the relations R_1 *and* R_2 are given by the tables

$R_1 =$

Y X	y_1	y_2	y_3
x_1	0	0.1	0.2
x_2	0	0.7	0.3
x_3	0.2	0.8	1.0

$R_2 =$

Y X	y_1	y_2	y_3
x_1	0.3	0.3	0.2
x_2	0.5	0	1.0
x_3	0.7	0.3	0.1

By using the above definitions we have in the following tables

$R_1 \cap R_2 =$

Y \ X	y_1	y_2	y_3
x_1	0	0.1	0.2
x_2	0	0	0.3
x_3	0.2	0.3	0.1

$R_1 \cup R_2 =$

Y \ X	y_1	y_2	y_3
x_1	0.3	0.3	0.2
x_2	0.5	0.7	1.0
x_3	0.7	0.8	1

Obviously, the proper inclusion $(R_1 \cap R_2) \subset (R_1 \cup R_2)$ holds.

4. Composition of fuzzy relations

The operation composition combines fuzzy relations in different variables (x, y) and (y, z) ; $x \in A$, $y \in B$, $z \in C$. Consider the relations

$$R_1(x, y) = \{\left((x, y), \quad \mu_{R_1}(x, y)\right)\} \quad , \quad (x, y) \in A \times B \, .$$

$$R_2(y, z) = \{\left((y, z), \mu_{R_2}(y, z)\right\} \quad , \quad (y, z) \in B \times C \, .$$

The domain of R_1 is $A \times B$ and the domain of R_2 is $\mathrm{B} \times C$.

The max-min composition denoted by $R_1 o R_2$ with membership function $\mu_{R_1 o R_2}$ is defined by

$$R_1 o R_2 = \{\left((x, z), \max\left(\min\left(\mu_{R_1}(x, y), \mu_{R_2}(y, z)\right)\right)\right)\} ,$$
$$(x, z) \in A \times C \, , y \in B \, .$$

Hence it is a relation in the domain $\mathrm{A} \times C$.

Example 4.1

Consider the relations $R_1(x, y)$ and $R_2(y, z)$ given as follows

$$R_1 = \{\left((x_1, y_1), 0.1\right), \left((x_1, y_2), 0.3\right), \left((x_1, y_3), 0\right), \left((x_2, y_1), 0.8\right), \left((x_2, y_2), 1\right), \left((x_2, y_3), 0.4\right)\}$$

$$R_2 = \{\left((y_1, z_1), 0.8\right), \left((y_1, z_2), 0.2\right), \left((y_1, z_3), 0\right), \left((y_2, z_1), 0.2\right), \left((y_2, z_2), 1\right), \left((y_2, z_3), 0.6\right), \left((y_3, z_1), 0.5\right), \left((y_3, z_2), 0\right), \left((y_3, z_3), 0.4\right)\}$$

To compute $R_1(x, y) o R_2(y, z)$ and perform the min operation that is

$$\min\left(\mu_{R_1}(x_1, y_1), \mu_{R_2}(y_1, z_1)\right) = \min(0.1, 0.8) = 0.1$$

$$\min\left(\mu_{R_1}(x_1, y_2), \mu_{R_2}(y_2, z_1)\right) = \min(0.3, 0.2) = 0.2$$

$$\min\left(\mu_{R_1}(x_1, y_3), \mu_{R_2}(y_3, z_1)\right) = \min(0, 0.5) = 0$$

Now we compute the max operation for $x = x_1, z = z_1, y = y_j$, $j = 1,2,3$.

$\{(x_1, z_1), \max(0.1, 0.2, 0)\} = \{(x_1, z_1), 0.2\}$,

So the grade of membership of the pair (x_1, z_1) is 0.2 . Hence the final result of the max-min composition is

$$R_1 o R_2 = \{\left((x_1, z_1), 0.2\right), \left((x_1, z_2), 0.3\right), \left((x_1, z_3), 0.3\right), \left((x_2, z_1), 0.8\right), \left((x_2, z_2), 1\right), \left((x_2, z_3), 0.6\right)\}.$$

4.2 Properties of composition of fuzzy relations

Let R, R_1, R_2, R_3 are fuzzy relations.

(i) $RoO = OoR \quad (O \rightarrow Zero\ relation)$
(ii) $RoI = IoR \quad (I \rightarrow Identity\ relation)$
(iii) $R_1 o R_2 \neq R_2 o R_1 (in\ general)$
(iv) $R_1 o(R_2 \cup R_3) = (R_1 o R_2) \cup (R_1 o R_3)$
(v) $R_1 o(R_2 \cap R_3) = (R_1 o R_2) \cap (R_1 o R_3)$
(vi) $R_2 \subseteq R_3 \Rightarrow (R_1 o R_2) \subseteq (R_1 o R_3)$
(vii) $R_1 o(R_2 o R_3) = (R_1 o R_2) o R_3$

5. Conclusion

Fuzzy relation established strength and association between elements of two sets .The membership values indicates linguistic relation like very far and much greater than etc. Here also, we established some properties and composition of fuzzy relations.

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