A STUDY ON RAMANUJAN FOURIER SERIES

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Abstract

Fourier was a mathematician in 1822. He gives Fourier series and Fourier transform to convert a signal into frequency domain. Fourier series simply states that, periodic signals can be represented into sum of sines and cosines when multiplied with a certain weight. We look at infinite series expansions for arithmetic functions, first considered by Srinivasa Ramanujan in 1918. A basis for these expansions is investigated, for which several properties are proven. Examples of these infinite series are established using multiple techniques.

Keywords: Arithmetic, Ramanujan, Fourier series.

Introduction

In 1918 the Indian mathematician Srinivasa Ramanujan published a paper titled "On Certain Trigonometrical Sums and their Applications in the Theory of Numbers"[16] in which he studied sums of the form

$$c_q(n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} 2\pi n \frac{a}{q}$$

where q and n are natural numbers and (a, q) is the greatest common divisor of a and q. More recently we consider equal sums, now called Ramanujan sums, of the form

$$c_q(n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e^{2\pi i n \frac{a}{q}}$$

and for cleanliness of notation, we will write e(t) = $e^{2\pi i \frac{a}{q}}$for t ∈ R.

An arithmetic function is a complex valued function defined on the set of natural numbers, the most useful of which express some number-theoretic property. Ramanujan used the sums cq(n) as a sort of basis to represent arithmetic functions as infinite series in a way that is analogous to the Fourier series representation of a function. It is this idea that will be the focus of this paper.

We will follow the reasoning of Gadiyar and Padma and use these infinite series as a tool to study the twin prime problem and generalizations. By choosing an appropriate arithmetic function and naively applying a natural analogue of the Wiener-Khinchin formula, we make a conjecture about the asymptotic value of the number of twin primes less than any value.

Finally, we use the infinite series representation of an arithmetic function to compute its values exactly. By truncating the sum after sufficiently many summands we are assured to be close enough to the actual value of the function that rounding the result will give us the correct value.

The properties of Ramanujan sums that we have proved now lead us to the main topic of the previously mentioned 1918 paper. In a way that is analogous to the Fourier series expansion of a function, Ramanujan used these sums as a basis for infinite series expansions for arithmetic functions.

Definition. Let a : N → C be an arithmetic function. Then a Ramanujan-Fourier series, or Ramanujan expansion, for the function a(n) is an infinite series of the form

$$a(n) = \sum_{q=1}^{\infty} a_q c_q(n)$$

Using elementary methods he was able to produce infinite series expansions for many of the commonly used arithmetic functions. A typical example is

$$\frac{\sigma(n)}{n} = \frac{\pi^2}{6} \sum_{q=1}^{\infty} \frac{c_q(n)}{q^2}$$

Where$\sigma(n) = \sum_{dln} d$

It is not true that Ramanujan expansions for all arithmetic functions can be found using elementary properties of infinite series and simple algebra. In fact the following expansion,

$$0 = \sum_{q=1}^{\infty} \frac{c_q(n)}{q}$$

is actually equivalent to the prime number theorem as we will see in chapter 4. This example also shows us that the Ramanujan expansion of a function is not unique in general, since the function that is identically zero also has the trivial expansion with all coefficients equal to zero.

Absent from Ramanujan's paper was a formula for the general coefficient in a Ramanujan expansion. A special case of this was done later in 1930 by Carmichael[3]. In that paper Carmichael also generalized Ramanujan's idea so that any arithmetic function with similar properties to cq(n) can be used in the same way as a basis for an infinite series expansion for another arithmetic function.

Carmichael's results were heavily based on the concept of the mean-value of an arithmetic function.

Definition: For an arithmetic function a(n), the limit

$$M(a) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a(n)$$

if it exists, is called the mean value of the function a.

Carmichael's discovery was a general formula for the coefficients of the Ramanujan expansion of an arithmetic function a(n) for which the mean value M (ac_q) exists for every q ∈ N. It is not true in general that the mean value M (ac_q) exists for every q ∈ N, but some progress has been made as to when we can be sure of existence.

We will require some preliminary definitions before those results can be stated. Additionally, the following results deal with two classes of arithmetic functions,namely additive and multiplicative functions. As we have seen before a multiplicative function is an arithmetic function a(n) for which

a(nm) = a(n)a(m)

whenever (n, m) = 1. An example of this is the Ramanujan sum cq(n) as was proved previously. An additive function is an arithmetic function for which

a(nm) = a(n) + a(m)

whenever (n, m) = 1. An example of an additive function is the restriction of the logarithm to the natural numbers, since

log(ab) = log(a) + log(b),

in this case for all integers a and b.

We will also define a semi-norm that will provide us with a crucial hypothesis for the upcoming results.

Definition: For an arithmetic function a(n) and 1 ≤ q < ∞, q ∈ R, define the semi-norm

$$\|a\|_q = \left(\lim_{x\to\infty} sup \frac{1}{x} \sum_{n\leq x} |a(n)|^q\right)^{1/q}$$

and

$$\|a\|_\infty = sup\{|a(n)|: n \in N\}$$

Note that when q = ∞ we actually have a norm above.

We will associate to an arithmetic function a(n) the Dirichlet series

$$\breve{a}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

absolutely convergent in the half-plane R(s) > k + 1 when the sequence$\{a_n\}_{n=1}^{\infty}$ satisfies $a_n = O(n^k)$.

For any prime p define the function a_p: N → C by

$$a_p(n) = \begin{cases} a(n) if n = p^k, k \in N \cup \{0\} \\ 0 \quad\quad otherwise \end{cases}$$

Also, for$l \in N \cup \{0\}$, define

$$\breve{a}_{p,l}(s) = \sum_{n=1}^{\infty} \frac{a(p^k)}{p^{ls}}$$

Finally we require one more definition before the results can be stated.

Definition: For two arithmetic functions a and b, define the Dirichlet convolution ofa and b, denoted a ∗ b, by

$$(a * b)(n) = \sum_{d|n} a(d) b\left(\frac{n}{d}\right)$$

We can now state our results that give us conditions for the existence of the coefficients of the Ramanujan expansion for certain arithmetic functions.

Theorem. For a multiplicative arithmetic function a(n), let $\gamma = \mu * a$ and suppose$\|a\|q< \infty$ for some q > 1. If M (a) exists, and is nonzero, then

$$M(a) = \prod_p \breve{\gamma}_p(1)$$

and a has a Ramanujan expansion with coefficients

$$a_q = M(a) \prod_{p^l \| q} \frac{\breve{\gamma}_p(1)}{\breve{\gamma}_p(1)}$$

where$p^l \parallel q$means that $p^l|q$but p^l ‡ n.

The formula for the mean value M (a) is due to Elliott, which was applied by Tuttas and Indlekofer to arrive at the formula for the coefficients of the Ramanujan expansion. The next theorem about additive functions is due to Hildebrand and Spilker.

Theorem. For an additive arithmetic function $a(n)$, let$\gamma = \mu * a$and suppose $\|a\|q <\infty$ for some q > 1. If M (a) exists, then

$$M(a) = \sum_p \breve{\gamma}_p(1)$$

and a has a Ramanujan expansion with coefficients

$$a_q = \frac{M(ac_q)}{\varphi(q)}$$

These methods of computing Ramanujan expansions rely on the mean-value of the arithmetic function, but there are plenty of arithmetic functions for which the mean-value does not exist so we are unable to apply the above results.

A 2010 paper by Lucht[15] introduced a new concept that would greatly increase the number of functions for which we are able to compute a Ramanujan expansion. His idea is largely based on the fact that cq(n) is closely related to the M¨obius function μ(q). His first result is useful in that not only can it help to determine the coefficients of a Ramanujan expansion, it can also be used to sum a Ramanujan expansion to find the arithmetic function it represents.

Conclusion

Fourier series are a powerful tool in applied mathematics; indeed, their importance is twofold since Fourier series are used to represent both periodic real functions as well as solutions admitted by linear partial differential equations with assigned initial and boundary conditions.

They play, in the case of regular periodic real functions, a role analogue to that one of Taylor polynomials when smooth real functions are considered. The idea of representation of a periodic function via a linear superposition of trigonometric functions finds, according to seminal origins back in Babylonian mathematics referring to celestial mechanics. Then, the idea was forgotten for centuries; thus, only in the eighteenth century,

Euler (1748) and, later, D. Bernoulli (1753) and Lagrange (1759), looking for solutions of the wave equation referring to a string with fixed extrema, introduce sums of trigonometric functions. However, a systematic study is due to Fourier who is the first to write a 2p-periodic function as the sum of a series of trigonometric functions. Specifically, trigonometric polynomials are introduced as a tool to provide an approximation of a periodic function. Historical notes on the subject are comprised in where the influence of Fourier series, whose introduction forced mathematicians to find an answer to many new questions, is pointed out emphasizing their relevance in the progress of Mathematics. Since the fundamental work by Fourier, Fourier series became a very well-known and widely used mathematical tool when representation of periodic functions is concerned. The aim of this section is to provide a concise introduction on the subject aiming to summarize those properties of Fourier series which are crucial under the applicative viewpoint. Indeed, the aim is to provide those notions which are required to apply Fourier series representation of periodic functions throughout the volume when needed. The interested reader is referred to specialized texts on the subject, such as to name a few of them. Accordingly, the Fourier theorem is stated with no proof. Conversely, its meaning is illustrated with some examples, and formulae are given to write explicitly the related Fourier series.

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