CONFIGURATION OF STREAMLINES OF A ROTATING FLUID FLOWS WITH VARIABLE VISCOSITY

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Abstract—A variably viscous fluid flow is considered flowing in *xy* – plane which is rotating about the *z* – axis with the uniform angular velocity. By using stream function approach the basic equations are written into convenient form. By the assumption that the streamlines are parallel straight lines, the geometry of magnetic lines and solutions to the flow variables are found and the variations of pressure and Coriolis force is plotted in graphs at various angular velocities.

I. INTRODUCTION

We consider a steady, incompressible fluid with variable viscosity, inclined variably in a rotating frame. The concept of Martin [5] is used in which the orders of the governing differential equations are reduced from higher to lower. A fluid motion is generally described by giving flow pattern. i.e. streamlines and magnetic lines. We study flows with the objective of obtaining exact solutions to various flow configurations.

II. MATHEMATICAL FORMULATION

A fluid flow is considered in xy – plane in which the coordinate system is rotating uniformly about the z – axis with uniform angular velocity $\vec{\omega}$. The governing equations for the steady flow in a rotating reference frame of a homogeneous incompressible viscous fluid with infinite electrical conductivity, are given as

$$div\vec{V}=0$$

$$\rho\left[\left(\vec{V}.\nabla\right)\vec{V}+2\vec{\omega}\times\vec{V}+\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)\right]=-\nabla p+\eta\nabla^2\vec{V}+\mu\vec{J}\times\vec{H}$$

$$\nabla\times\left(\vec{V}\times\vec{H}\right)=0$$

$$div\,\vec{H}=0$$

where $\vec{V}=(u,v)$ = velocity vector, $\vec{H}=(H_1,H_2)$ = magnetic field vector, ρ = fluid density, $\vec{\omega}$ = angular velocity, r = radius vector, p = fluid pressure, η = coefficient of viscosity, μ = magnetic permeability, $\vec{J}=\nabla\times\vec{H}$ = current density vector.

In this paper, the basic equations of motion governing steady flow in a rotating reference frame of a homogeneous incompressible fluid with variable viscosity and with infinite electrical conductivity, are reformulated as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$\eta\frac{\partial\xi}{\partial y}-\rho v\xi-2\frac{\partial u}{\partial x}\frac{\partial\eta}{\partial x}-\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\frac{\partial\eta}{\partial y}-2\rho\omega v+\mu jH_2=-\frac{\partial E}{\partial x} \tag{2}$$

$$\eta\frac{\partial\xi}{\partial x}-\rho u\xi-2\frac{\partial v}{\partial y}\frac{\partial\eta}{\partial y}+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\frac{\partial\eta}{\partial x}-2\rho\omega u+\mu jH_1=\frac{\partial E}{\partial y} \tag{3}$$

$$uH_2-vH_1=c \tag{4}$$

$$\frac{\partial H_1}{\partial x}+\frac{\partial H_2}{\partial y}=0 \tag{5}$$

$$\xi=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} \tag{6}$$

$$j=\frac{\partial H_2}{\partial x}-\frac{\partial H_1}{\partial y} \tag{7}$$

$$E=\frac{1}{2}\rho\left(u^2+v^2\right)+p \tag{8}$$

where $V^2=u^2+v^2$ and $H^2=H_1^2+H_2^2$, c is the diffusion constant. Equations (6), (7) and (8) give vorticity ξ, current density j and energy function E respectively.

Equations (1) to (8) form a system of partial differential equations in eight unknowns u, v, H_1, H_2, ξ, j, p and E.

Now we consider variably inclined plane flows and let α (x, y) be the angle of inclination, then using diffusion equation and (4), we get

$$uH_2-vH_1=VH\sin\alpha=c \tag{9}$$

$$uH_1+vH_2=VH\cos\alpha=c\cot\alpha \tag{10}$$

Solving (9) and (10) for H_1 and H_2, we get

$$H_1=c\left(\frac{u\cot\alpha-v}{V^2}\right) \tag{11}$$

$$H_2=c\left(\frac{u+v\cot\alpha}{V^2}\right) \tag{12}$$

Eliminating H_1 and H_2 from (1) to (8) using (11) and (12), we obtain

$$\eta\frac{\partial\xi}{\partial y}-\rho v\xi-2\frac{\partial u}{\partial x}\frac{\partial\eta}{\partial x}-\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\frac{\partial\eta}{\partial y}-2\rho\omega v+\mu jc\left(\frac{u+v\cot\alpha}{V^2}\right)=-\frac{\partial E}{\partial x} \tag{13}$$

$$\eta\frac{\partial\xi}{\partial x}-\rho u\xi-2\frac{\partial v}{\partial y}\frac{\partial\eta}{\partial y}+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\frac{\partial\eta}{\partial x}-2\rho\omega u+\mu jc\left(\frac{u\cot\alpha-v}{V^2}\right)=\frac{\partial E}{\partial y} \tag{14}$$

$$\frac{\partial}{\partial x}\left(\frac{u\cot\alpha-v}{V^2}\right)+\frac{\partial}{\partial y}\left(\frac{u+v\cot\alpha}{V^2}\right)=0 \tag{15}$$

$$\frac{\partial}{\partial x}\left(\frac{u+v\cot\alpha}{V^2}\right)-\frac{\partial}{\partial y}\left(\frac{u\cot\alpha-v}{V^2}\right)=\frac{j}{c} \tag{16}$$

With unknowns u, v, ξ, j, E and α (x, y), once the solution of this system is determined, the pressure p and the magnetic field H are obtained from (8), (11) and (12).

Now the equation of continuity (1) implies the existence of a stream function ψ (x, y) such that

$$d\psi=\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy$$

i.e. $d\psi=-vdx+udy$ where $\frac{\partial\psi}{\partial x}=-v,\ \frac{\partial\psi}{\partial y}=u$

From (1) we get

$$\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial\psi}{\partial x}\right)=\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial^2\psi}{\partial y\partial x}=0$$

This justifies the above substitution.

Now substituting u and v, we get from (13)

$$\eta\frac{\partial\xi}{\partial y}+\rho\xi\frac{\partial\psi}{\partial x}-2\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial\eta}{\partial x}-\left(\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right)\frac{\partial\eta}{\partial y}+2\rho\omega\frac{\partial\psi}{\partial x}+\mu jc\frac{\left(\frac{\partial\psi}{\partial y}-\frac{\partial\psi}{\partial x}\cot\alpha\right)}{\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2}=-\frac{\partial E}{\partial x} \tag{17}$$

Similarly (14) gives

$$\eta\frac{\partial\xi}{\partial x}-\rho\xi\frac{\partial\psi}{\partial y}+2\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial\eta}{\partial y}+\left(\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right)\frac{\partial\eta}{\partial x}-2\rho\omega\frac{\partial\psi}{\partial y}+\mu jc\frac{\left(\frac{\partial\psi}{\partial y}\cot\alpha+\frac{\partial\psi}{\partial x}\right)}{\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2}=\frac{\partial E}{\partial y} \tag{18}$$

(15) gives

$$\frac{\partial}{\partial x}\left[\frac{\frac{\partial\psi}{\partial y}\cot\alpha+\frac{\partial\psi}{\partial x}}{\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2}\right]+\frac{\partial}{\partial y}\left[\frac{\frac{\partial\psi}{\partial y}-\frac{\partial\psi}{\partial x}\cot\alpha}{\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2}\right]=0 \tag{19}$$

(6) gives $$\xi=-\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2} \tag{20}$$

(16) gives

$$\frac{\partial}{\partial x}\left[\frac{\frac{\partial\psi}{\partial y}-\frac{\partial\psi}{\partial x}\cot\alpha}{\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2}\right]-\frac{\partial}{\partial y}\left[\frac{\frac{\partial\psi}{\partial y}\cot\alpha+\frac{\partial\psi}{\partial x}}{\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2}\right]=\frac{j}{c} \tag{21}$$

Now $\frac{\partial}{\partial y}(17)+\frac{\partial}{\partial x}(18)$ gives

$$\begin{aligned}
&\frac{\partial\eta}{\partial y}\frac{\partial\xi}{\partial y}+\frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial x}+\eta\left(\frac{\partial^2\xi}{\partial y^2}+\frac{\partial^2\xi}{\partial x^2}\right)+\\
&\rho\left[\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}(\xi+2\omega)-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\xi+2\omega)\right]\\
&+4\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^2\eta}{\partial x\partial y}+2\left[\frac{\partial}{\partial y}\left(\frac{\partial^2\psi}{\partial x\partial y}\right)\frac{\partial\eta}{\partial x}+\frac{\partial}{\partial x}\left(\frac{\partial^2\psi}{\partial x\partial y}\right)\frac{\partial\eta}{\partial y}\right]\\
&+\left[\frac{\partial\eta}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right)+\frac{\partial\eta}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right)\right]\\
&-\left(\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right)\left(\frac{\partial^2\eta}{\partial y^2}-\frac{\partial^2\eta}{\partial x^2}\right)\\
&+\frac{\mu c}{\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2}\left\{\left(\frac{\partial\psi}{\partial y}-\frac{\partial\psi}{\partial x}\cot\alpha\right)\frac{\partial j}{\partial y}\right. \qquad (22)\\
&\left.+\left(\frac{\partial\psi}{\partial y}\cot\alpha+\frac{\partial\psi}{\partial x}\right)\frac{\partial j}{\partial x}\right\}=0
\end{aligned}$$

Now we study the geometry of magnetic lines and the solutions of velocity components, angular velocity and pressure function when streamlines are straight lines and parallel. Here magnetic lines are variably inclined but nowhere aligned with the streamlines, so for constantly inclined flows we take α as constant.

We now assume that

$$\psi(x,y)=S(y), \qquad S'(y)\neq 0$$
$$\eta(x,y)=T(y), \qquad T'(y)\neq 0$$

Now from (20) we get

$$\xi=-S''(y)$$

And from (20) we get

$$-c\frac{\partial}{\partial y}\left[\frac{\cot\alpha}{S'(y)}\right]=j$$

Then (22) becomes

$$\eta S'S^{iv}+T''S'S''+\mu c^2\left(\frac{\partial^2}{\partial y^2}\left[\frac{\cot\alpha}{S'}\right]+\cot\alpha\frac{\partial^2}{\partial x\partial y}\left[\frac{\cot\alpha}{S'}\right]\right)=0$$

where

$$\frac{\partial^2}{\partial y^2}\left(\frac{\cot\alpha}{S'}\right)=\left(2\frac{S''^3}{S'^3}-2\frac{S''S'''}{S'^2}-\frac{S'''}{S'^2}+\frac{S^{iv}}{S'}\right)x$$
$$+S'''g+2S''g''+S'g'$$

$$\frac{\partial^2}{\partial x\partial y}\left(\frac{\cot\alpha}{S'}\right)=\frac{S'S'''-2S''S'''}{S'^3}$$

$$\eta S'S^{iv}+T''S'S''+\mu c^2\left(\frac{\partial^2}{\partial y^2}\left[\frac{\cot\alpha}{S'}\right]+\cot\alpha\frac{\partial^2}{\partial x\partial y}\left[\frac{\cot\alpha}{S'}\right]\right)=0$$

Finally we consider,

$S''(y)=0, T(y)=0$ and $G(y)=0$

Then $\psi=S(y)=N_1y+N_2$, $G(y)=N_3y+N_4$ and $N_1\neq 0$, N_2, N_3, N_4 are arbitrary constants.

Using above result for stream function, we now find

$$v_1=\frac{\partial\psi}{\partial y}=S'(y)=N_1, \qquad v_2=\frac{-\partial\psi}{\partial x}=0$$

$$\cot\alpha=N_3y+N_4=G(y)$$

$$\xi=-S''(y)=0$$

$$j=-c\frac{\partial}{\partial y}\left(\frac{\cot\alpha}{S'}\right)=-c\frac{\partial}{\partial y}\left(\frac{N_3y+N_4}{N_1}\right)=-c\frac{N_3}{N_1}$$

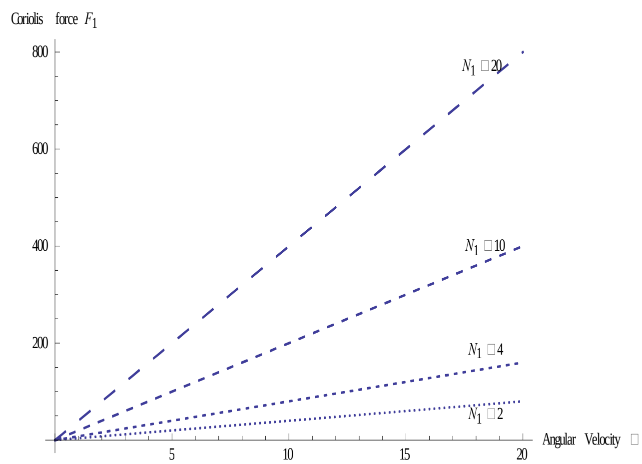
$$H_1=c\left\{\frac{N_1(N_3y+N_4)}{N_1^2}\right\}=\frac{c}{N_1}(N_3y+N_4)$$

$$H_2=c\left(\frac{N_1}{N_1^2}\right)=\frac{c}{N_1}$$

The magnitude of the Coriolis force is

$F_1=2\omega N_1$

The graph is plotted below:



Graph of coriolis force with respect to angular velocity

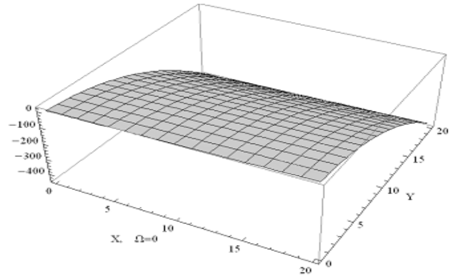
Using (2) and (3) we find

$$E=\frac{\mu c^2N_3}{N_1^2}\left(x-\frac{N_3y^2}{2}-N_4y\right)-2\rho\omega N_1y+N_0 \qquad \text{where}$$

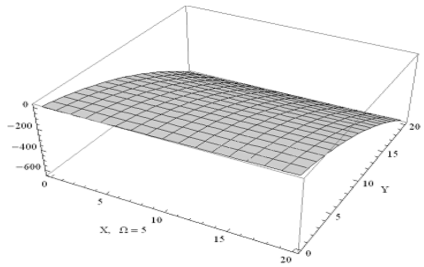
N_0 is a constant and hence the pressure function is

$$p(x,y)=\frac{\mu c^2N_3}{N_1^2}\left(x-\frac{N_3y^2}{2}-N_4y\right)-\frac{1}{2}\rho N_1^2y-2\rho N\omega y$$

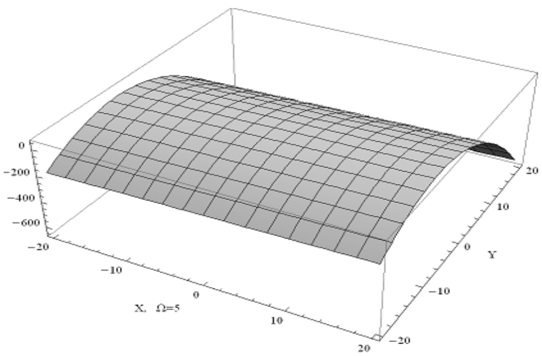
The variation of this p for different angular velocities is shown in figure below



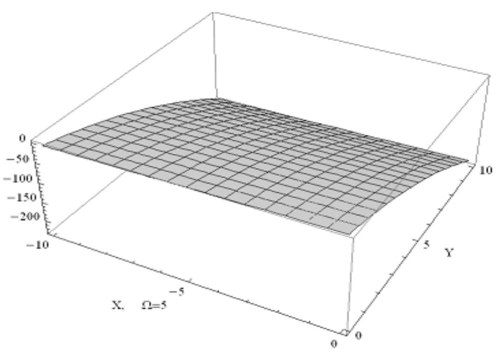
Plot the graph of the variation of p for different angular velocities



Plot the graph of the variation of p for different angular velocities

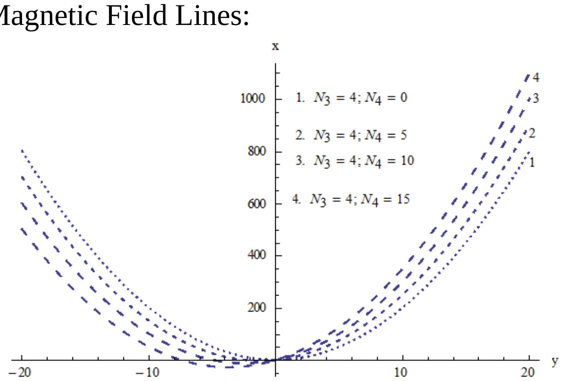


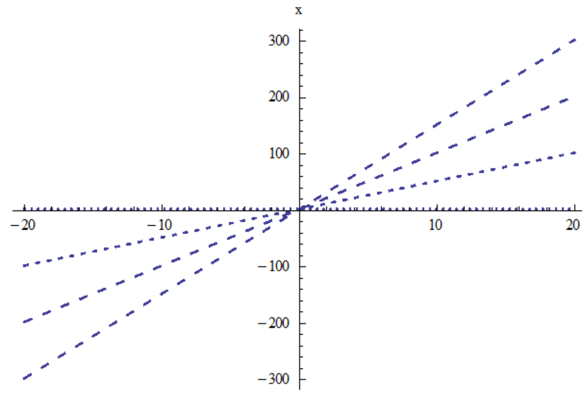
Plot the graph of the variation of p for different angular velocities



Plot the graph of the variation of p for different angular velocities

Magnetic Field Lines:





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