A QUARTIC LEGENDRE SPLINE COLLOCATION METHOD TO SOLVE FREDHOLM INTEGRO DIFFERENTIAL EQUATION

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Abstract
In this paper, a new approach to solve Fredholm integro differential equation has been introduced. Quartic Legendre spline collocation technique is used to solve Fredholm integro differential equation. Using this novel approach we have tried to achieve more accuracy compare to other traditional methods. Using the prescribed method the solution of this problem is reduce to a linear system of equations. The results demonstrate the applicability and accuracy of the technique.

Index Terms: Quartic Legendre polynomial, spline collocation, Integro differential equation

I. INTRODUCTION

Integro differential equations appears in various physical problems in sciences and engineering. It can be find useful in a wide range of application fields such as Computer graphics, Image processing, Biological problems and financial problems. Therefore, their numerical solutions are very useful to analyse the problems related to various fields. But it is usually becomes difficult to solve Integro differential equations analytically so there is an always a need to find an approximate solution which may be efficient and accurate.

Many attempt has been done to solve integro differential equations before, like Homotopy perturbation methods [1], [2], New Homotopy Analysis method (NHAM) [3], Wavelet-Galerkin method [4], Differential transform method(DTM) [5], Adomain decomposition method [6], CAS wavelets method [7], Sine-Cosine wavelets [2].

Spline function is very useful function in getting smoother approximations of a various problems of engineering and higher mathematics. It has a good ability of approximation. Various authors has been used spline function since last decade for solving differential equation, integral equation, integro differential equation, partial differential equation and fractional differential equation.

There is not a well expressed orthogonal basis in spline space to date [8]. Here we have constructed an orthogonal basis for the nth degree spline space with the help of Legendre polynomial. Hence we have extended idea of using Legendre spline function instead of spline function for approximating solution of second order integro differential equations of the form:

$$y''(x) + y'(x) + f(x)y(x) + \lambda \int_a^b k(x,s)y(s)ds = g(x), \; y(a) = y_0 \qquad (1.1)$$

where the functions f(x), g(x) and k(x, s) are sufficiently smooth real valued functions.

The remainder of this paper is organized as follows: In Section 2, we describe the basic formulation of Quartic Legendre spline collocation method required for our subsequent development. In section 3 description of the method is given and Section 4 is devoted to the applicability and accuracy of propose method by considering numerical examples. Also a conclusion is given in the last Section.

II. QUARTIC LEGENDRE SPLINE POLYNOMIAL

Let u(x) be a function defined on an interval [a, b] and $\Delta : a = x_0 < x_1 < \cdots\ldots\ldots < x_N = b$, be a given partition of the interval [a, b]. Each point x_i (i = 0 (1) N) is also called a knot or node, for the space of spline functions.

A polynomial function $s(x) \in C^2[a,b]$, which interpolates u(x) at the mesh points x_i, $i = 0(1)N$ is called a Quartic Legendre spline, if it coincides with a Quartic Legendre polynomial $s_i(x)$ in each subinterval $[x_i, x_{i+1}]$. Let u_i be an approximation to $u(x_i)$ obtained by a segment $s_i(x)$ passing through points (x_i, u_i) and (x_{i+1}, u_{i+1})

For each segment $[x_i, x_{i+1}]$, $s_i(x)$ has the following form

$$s_i(x) = \frac{1}{8}a_i[35(x-x_i)^4 - 30(x-x_i)^2 + 3] + \frac{1}{2}b_i[5(x-x_i)^3 - 3(x-x_i)] + \frac{1}{2}c_i[3(x-x_i)^2 - 1] + d(x-x_i) + e_i \quad (2.1)$$

Where, $s_i(x_i) = y_i$, $s_i(x_{i+1}) = y_{i+1}$, $s_i''(x_i) = M_i$, $s_i''(x_{i+1}) = M_{i+1}$, $s_i'''(x_i) = F_i$, $s_i'''(x_{i+1}) = F_{i+1}$ and $s_i^{iv}(x_i) = \frac{F_i + F_{i+1}}{2}$

$$a_i = \frac{F_i + F_{i+1}}{210}$$

$$b_i = \frac{M_{i+1} - M_i}{15h} - \frac{h}{60}(F_{i+1} + F_i)$$

$$c_i = \frac{M_i}{3} + \frac{F_{i+1} + F_i}{84}$$

$$d_i = \frac{1}{h}(y_{i+1} - y_i) + \frac{h^3}{48}(F_{i+1} + F_i) - \frac{h}{6}M_{i+1} - \frac{h}{3}M_i + \frac{M_{i+1}}{10h} + \frac{M_i}{10h} + \frac{h}{40}(F_{i+1} + F_i)$$

$$e_i = y_i + \frac{1}{6}M_i + \frac{1}{240}(F_{i+1} + F_i)$$

III. DESCRIPTION OF THE METHOD

To develop the Quartic Legendre spline collocation method for solving the integro-differential equation (1.1) the interval [a, b] is divided into n equal subintervals using the grids i = 0, 1, …… ,n where $h = \frac{b-a}{N}$, the Quartic Legendre Spline S(x) interpolating the functions y(x) at the grid points is given by the equation

$$s_i(x) = \frac{F_{i+1} + F_i}{48}(x-x_i)^4 + \left(\frac{M_{i+1} - M_i}{6h} - \frac{h}{24}(F_{i+1} + F_i)\right)(x-x_i)^3 + \frac{M_i}{2}(x-x_i)^2 + \begin{bmatrix} \frac{h^3}{48}(F_{i+1} + F_i) - \frac{h}{6}(M_{i+1} + 2M_i) \\ + \frac{1}{h}(y_{i+1} - y_i) \end{bmatrix}(x-x_i) + y_i \quad (3.1)$$

Where $M_i = s''(x_i)$ and $y_i = y(x_i)$. The unknown derivative M_i are related by enforcing the continuity condition on $s'(x)$. Differentiating (3.1), we obtain

$$s_i'(x) = \frac{F_{i+1} + F_i}{12}(x-x_i)^3 + \left(\frac{M_{i+1} - M_i}{2h} - \frac{h}{8}(F_{i+1} + F_i)\right)(x-x_i)^2 + M_i(x-x_i) + \begin{bmatrix} \frac{h^3}{48}(F_{i+1} + F_i) - \frac{h}{6}(M_{i+1} + 2M_i) \\ + \frac{1}{h}(y_{i+1} - y_i) \end{bmatrix} \quad (3.2)$$

$$s_i''(x) = \frac{F_{i+1} + F_i}{4}(x-x_i)^2 + \left(\frac{M_{i+1} - M_i}{h} - \frac{h}{4}(F_{i+1} + F_i)\right)(x-x_i) + M_i \quad (3.3)$$

$$s_i'''(x) = \frac{F_{i+1} + F_i}{2}(x-x_i) + \left(\frac{M_{i+1} - M_i}{h} - \frac{h}{4}(F_{i+1} + F_i)\right) \quad (3.4)$$

We obtain one sided limits of the derivative from (3.2) as

$$s_i'(x_i^-) = \begin{bmatrix} -\frac{h^3}{48}(F_i + F_{i-1}) \\ + \frac{h}{6}(2M_i + M_{i-1}) \\ + \frac{1}{h}(y_i - y_{i-1}) \end{bmatrix}, i = 1, 2, \ldots, n \quad (3.5)$$

$$s_i'(x_i^+) = \frac{h^3}{48}(F_{i+1} + F_i) - \frac{h}{6}(M_{i+1} + 2M_i) + \frac{1}{h}(y_{i+1} - y_i), \; i = 0, 1, 2, \ldots\ldots, n-1 \quad (3.6)$$

Similarly we obtain from equation (3.3) that

$$s_i''(x_i^-) = [M_i], \quad i = 1, 2, \ldots\ldots, n \quad (3.7)$$

And

$$s_i''(x_i^+) = [M_i], \quad i = 0, 1, \ldots\ldots, n-1 \quad (3.8)$$

Also from equation (3.4) we obtain

$$s_i'''(x_i^-) = \frac{M_i - M_{i-1}}{h} + \frac{h}{4}(F_i + F_{i-1}),$$

$$i = 1, 2, \ldots\ldots, n \quad (3.9)$$

and

$$s_i'''(x_i^+) = \left(\frac{M_{i+1}-M_i}{h} - \frac{h}{4}(F_{i+1}+F_i)\right), \quad i = 0,1,\dots\dots,n-1 \qquad (3.10)$$

The continuity condition $s'(x_i^+) = s'(x_i^-)$ gives the consistency relation

$$M_{i+1} + 4M_i + M_{i-1} = \frac{6}{h^2}[y_{i+1} - 2y_i + y_{i-1}] + \frac{h^2}{8}[F_{i+1} + 2F_i + F_{i-1}], \quad i = 1(1)N-1 \qquad (3.11)$$

The continuity condition $s'''(x_i^+) = s'''(x_i^-)$ gives the consistency relation

$$M_{i+1} - 2M_i + M_{i-1} = \frac{h^2}{4}[F_{i+1} + 2F_i + F_{i-1}], \quad i = 1(1)N-1 \qquad (3.12)$$

Now we collocate equation (1.1) at the uniform grid points $x_i = x_0 + ih,\ i = 0,1,\dots\dots,n$ with $x_0 = a$ and $x_n = b$

Thus (1.1) becomes

$$y''(x_i) + y'(x_i) + f(x_i)y(x_i) + \lambda \sum_{j=0}^{n-1} \int_{s_j}^{s_{j+1}} k(x_i,s)y(s)\,ds = g(x_i), \quad i = 0,1,\dots\dots,n$$

Using equation (3.1) it can be written as

$$y''(x_i) + y'(x_i) + f(x_i)y(x_i) + \lambda \sum_{j=0}^{n-1} \int_{s_j}^{s_{j+1}} k(x_i,s) \left\{ \begin{array}{c} \frac{F_{j+1}+F_j}{48}(s-s_j)^4 \\ + \left(\begin{array}{c} \frac{M_{j+1}-M_j}{6h} \\ -\frac{h}{24}(F_{j+1}+F_j) \end{array} \right)(s-s_j)^3 \\ + \frac{M_j}{2}(s-s_j)^2 \\ + \left[\begin{array}{c} \frac{h^3}{48}(F_{j+1}+F_j) \\ -\frac{h}{6}(M_{j+1}+2M_j) \\ + \frac{1}{h}(y_{j+1}-y_j) \end{array} \right](s-s_j) \\ + y_j \end{array} \right\} ds = g(x_i), \quad i = 0,1,\dots\dots,n$$

Now substituting $s = s_j + ph$ and simplifying

$$y''(x_i) + y'(x_i) + f(x_i)y(x_i) + h\lambda \sum_{j=0}^{n-1} \int_0^1 k(x_i, s_j + ph) \left\{ \begin{array}{c} [\frac{F_{j+1}+F_j}{48}(h)^4 \\ *(p^4 - 2p^3 + p)] \\ + \frac{h^2}{6}(p^3 - p)M_{j+1} \\ + \frac{h^2}{6}(-p^3 + 3p^2 - 2p)M_j \\ + p y_{j+1} + (1-p)y_j \end{array} \right\} dp = g(x_i), \quad i = 0,1,\dots\dots,n$$

We now split above equation into two as

$$y''(x_0) + y'(x_0) + f(x_0)y(x_0) + h\lambda \sum_{j=0}^{n-1} \int_0^1 k(x_0, s_j + ph) \left\{ \begin{array}{c} [\frac{F_{j+1}+F_j}{48}(h)^4 \\ *(p^4 - 2p^3 + p)] \\ + \frac{h^2}{6}(p^3 - p)M_{j+1} \\ + \frac{h^2}{6}(-p^3 + 3p^2 - 2p)M_j \\ + p y_{j+1} + (1-p)y_j \end{array} \right\} dp = g(x_0) \qquad (3.13)$$

And

$$y''(x_i) + y'(x_i) + f(x_i)y(x_i) + h\lambda \sum_{j=0}^{n-1} \int_0^1 k(x_i, s_j + ph) \left\{ \begin{array}{c} [\frac{F_{j+1}+F_j}{48}(h)^4 \\ *(p^4 - 2p^3 + p)] \\ + \frac{h^2}{6}(p^3 - p)M_{j+1} \\ + \frac{h^2}{6}(-p^3 + 3p^2 - 2p)M_j \\ + p y_{j+1} + (1-p)y_j \end{array} \right\} dp = g(x_i), \quad i = 1,2,\dots,n \qquad (3.14)$$

Putting (3.6) and (3.8) for i = 0 in equation (3.13) we get

$$M_0 + \frac{h^3}{48}(F_1 + F_0) - \frac{h}{6}(M_1 + 2M_0) + \frac{1}{h}(y_1 - y_0) + f_0 y_0 + h\lambda \sum_{j=0}^{n-1} \int_0^1 k(x_0, s_j + ph) \left\{ \begin{array}{c} [\frac{F_{j+1}+F_j}{48}(h)^4 \\ *(p^4 - 2p^3 + p)] \\ + \frac{h^2}{6}(p^3 - p)M_{j+1} \\ + \frac{h^2}{6}(-p^3 + 3p^2 - 2p)M_j \\ + p y_{j+1} + (1-p)y_j \end{array} \right\} dp = g(x_0) \qquad (3.15)$$

Similarly putting (3.5) and (3.7) in (3.15) we get

$$M_i - \frac{h^3}{48}(F_i + F_{i-1}) + \frac{h}{6}(M_i + 2M_{i-1}) + \frac{1}{h}(y_i - y_{i-1}) + f_i y_i + h\lambda \sum_{j=0}^{n-1} \int_0^1 k(x_i, s_j + ph) \left\{ \begin{array}{c} [\frac{F_{j+1}+F_j}{48}(h)^4 \\ *(p^4 - 2p^3 + p)] \\ + \frac{h^2}{6}(p^3 - p)M_{j+1} \\ + \frac{h^2}{6}(-p^3 + 3p^2 - 2p)M_j \\ + p y_{j+1} + (1-p)y_j \end{array} \right\} dp = g(x_i), \quad i = 1,\dots,n \qquad (3.16)$$

Using equations (3.15) and (3.16) together with equation (3.11) and (3.12) consists of 2n+1 equations with 4(n) +1 unknowns $y_i, M_i, and\ F_i \quad i = 0,1,\dots\dots,n.$

However, to determine the values of these unknowns, four more equations are required.

These equations are obtained by using the initial condition of equation (1.1) and by imposing a free boundary conditions given below

$F_0 = 0$ And $F_2 = 0$ (3.17)

IV. NUMERICAL EXPERIMENT

In this section the propose method described in section 3 is applied to some examples of second order integro differential equations. All computations are carried out by using MATLAB.

Example 4.1: Consider the following IDE [N. Ebrahimi et al (2014)[11]], H.M.Jaradat et al (2009)[12], Ahmet Yildirim (2009)[13]]

$$y''(x) = e^{x} - x + \int_0^1 xty(t)dt; \quad y(0) = 1, y'(0) = 1 \qquad (4.1)$$

Solution:

Using the Quartic LSC method and with the help of equations (3.11), (3.12) and $F_0 = 0$, $F_2 = 0$, equation (4.1) is reduce to system of equations and which are being solved using gauss elimination method gives the unknowns

$M_1 = 2.3415 \quad M_2 = 2.1761$

$F_1 = -12.0556 \quad y_1 = 1.6409 \quad y_2 = 2.7156$

Repeat the same process for h = 1/3, we get the following values of unknowns

M1=1.4005; F1=3.0840; F2=-0.2314;
M0=1.3382; M2=1.9575; M3=2.7329;
u1=1.4080; u2=1.9793; u3=2.7715

Table (I a) presents the comparison of solution of our method with the exact solution and Table (I b) presents the comparison of absolute error of our method with the method implemented in [11].

I a: Quartic LSC solution of second order Fredholm IDE (4.1)

x:	Quartic LSC solution : h=1/2	Quartic LSC Solution : h=1/3	Exact Solution : $y(x)=e^x$
0	1	1	1
0.1	1.0942	1.1066	1.1052
0.2	1.2002	1.2265	1.2214
0.3	1.3193	1.3496	1.3499
0.4	1.4522	1.5079	1.4918
0.5	1.6409	1.6494	1.6487
0.6	1.8075	1.8251	1.8221
0.7	1.9986	2.0173	2.0138
0.8	2.2147	2.2244	2.2255
0.9	2.4557	2.4501	2.4596
1.0	2.7210	2.7187	2.7183

I b: Absolute errors of Quartic LSC solution of Fredholm IDE (4.1)

x:	Absolute Error Quartic SLC: h=1/2	Absolute error Quartic SLC: h=1/3	Absolute Error : [11] N=10	Absolute Error : [11] N=30
0	0	0	0	0
0.1	1.1×10^{-2}	1.4×10^{-3}	4.39×10^{-6}	4.82×10^{-7}
0.2	2.1×10^{-2}	5.1×10^{-3}	1.81×10^{-5}	2.0×10^{-6}
0.3	3.06×10^{-2}	3.0×10^{-4}	4.24×10^{-5}	4.69×10^{-6}
0.4	3.96×10^{-2}	1.61×10^{-2}	7.83×10^{-5}	8.67×10^{-6}
0.5	7.8×10^{-3}	7.0×10^{-4}	1.27×10^{-4}	1.41×10^{-5}
0.6	1.46×10^{-2}	3.0×10^{-3}	1.91×10^{-4}	2.11×10^{-5}
0.7	1.52×10^{-2}	3.5×10^{-3}	2.70×10^{-4}	2.99×10^{-5}
0.8	1.08×10^{-2}	1.1×10^{-3}	3.67×10^{-4}	4.07×10^{-5}
0.9	3.9×10^{-3}	9.5×10^{-3}	4.85×10^{-4}	5.37×10^{-5}
1.0	2.7×10^{-3}	4.0×10^{-4}	6.25×10^{-4}	6.91×10^{-5}

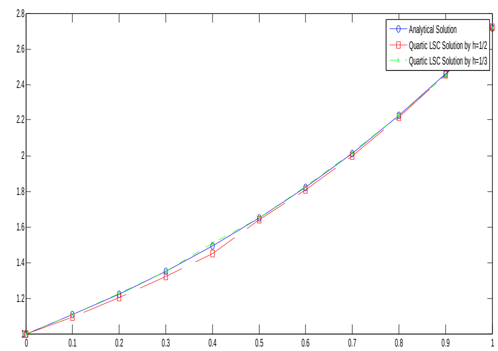

Fig. 4.1: A comparison of solutions of IDE (4.1)

Example 4.2: Consider the following IDE [A.A.Hemeda (2012)[14]]

$$y''(x) = 2 - \left(\frac{2x}{3}\right) + \int_0^1 xty'(t)dt; \quad y(0) = 0, y'(0) = 0 \qquad (4.2)$$

Solution:

The collocation equations are

$$M_0 = 2$$

$$2y_1 - \frac{M_1}{12} - \frac{M_0}{6} = 0$$

$$\frac{755M_0}{1536} + \frac{F_1}{7680} + \frac{F_2}{15360} + \frac{F_0}{15360} + \frac{187M_1}{384} + \frac{17M_2}{1536} + \frac{y_1}{4} - \frac{3y_2}{8} = \frac{5}{3}$$

$$-\frac{13M_0}{768} + \frac{F_1}{3840} + \frac{F_2}{7680} + \frac{F_0}{7680} + \frac{91M_1}{192} + \frac{401M_2}{768} + \frac{y_1}{2} - \frac{3y_2}{4} = \frac{4}{3}$$

We get the unknowns as

$M_1 = 2 \quad M_2 = 2 \quad F_1 = 0 \quad y_1 = 0.25 \quad y_2 = 1.0$

And solutions are shown in the following table

II: Quartic LSC solution of second order Fredholm IDE (4.2)

x:	Quartic LSC Solution:	Exact Solution: x^2
0	0	0
0.1	0.01	0.01
0.2	0.04	0.04
0.3	0.09	0.09
0.4	0.16	0.16
0.5	0.25	0.25
0.6	0.36	0.36
0.7	0.49	0.49
0.8	0.64	0.64
0.9	0.81	0.81
1.0	1	1

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