STREAMLINES IN CONSTANTLY INCLINED TWO – PHASE EMFD FLOWS

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Abstract
Streamlines of steady, plane, viscous, incompressible constantly inclined two – phase EMFD flows is calculated by solving a non – homogeneous partial differential equation of second order which is obtained by employing Hodograph transformation to the flow.

Index Terms: Electromagnetic fluid dynamics, two phase fluid, streamlines, number density.

I. INTRODUCTION

A vast amount of research is been done on the motion of electrically conducting fluids using MHD approximation since Alfven's [1] classic work. Chandana. et. al. [2] studied the incompressible fluids in the case of non – zero charge density without assuming the proportionality between the magnetic field and the aligned velocity field. Wan-Les-Yin [3] studied EMFD flows with non-zero charge density and analyzed that all steady incompressible aligned MFD plane flows are irrotational motions. Such irrotational flows are either steady rigid translational motions or radial flows symmetric with respect to singular point and in these flows the magnetic field is a constant multiple of the velocity field. Therefore for incompressible fluids with non-zero charge density, aligned field solutions having non proportional magnetic and velocity field do not exist, nor there exist any rotational flows.

On the other hand multiphase fluid phenomena are another extremely important field of science and technology such as geophysics, nuclear engineering, chemical engineering etc [4]. In recent years, considerable attention has been paid to the study of the multiphase fluid flow system in non-rotating or rotating frames of reference. Multiphase fluid systems are concerned with the motion of a fluid containing immiscible inert particles. Even though the presence of particles in fluid makes the dynamical study of flow problems very complicated, but many simplifying assumptions are made to investigate these problems.

Saffman [5] formulated the equations of motion of a dusty fluid which is represented in terms of large number density $N(x, y)$ of very small spherical inert particles whose volume concentration is small enough to be neglected. It is assumed that the density of the dust particle is large when compared with the fluid density so that the mass concentration of the particles is an appreciable fraction of unity.

II. BASIC EQUATIONS

If the electric charge density q (x, y) is a non – zero function, the governing equations for steady flow of a dusty, viscous, incompressible electro magneto fluid – dynamic flow are given by the following system:

$$div\,\vec{u} = 0 \quad (2.1)$$

$$div\,\vec{H} = 0 \quad (2.2)$$

$$curl\left(\vec{u} \times \vec{H}\right) = \vec{0} \quad (2.3)$$

$$curl\left(q\,\vec{u}\right) = \vec{0} \quad (2.4)$$

$$div\left(q\,\vec{u}\right)=-\frac{\sigma\, q}{\varepsilon} \tag{2.5}$$

$$\rho\left[\left(\vec{u}.\nabla\right)\vec{u}\right]=-\nabla p+\mu\left(\nabla\times\vec{H}\right)\times\vec{H}+KN\left(\vec{v}-\vec{u}\right)+\eta\nabla^2\vec{u}-\frac{q^2\vec{u}}{\sigma} \tag{2.6}$$

$$div\left(N\,\vec{v}\right)=0 \tag{2.7}$$

$$m\left(\vec{v}.grad\right)\vec{v}=K\left(\vec{u}-\vec{v}\right) \tag{2.8}$$

where $\vec{u}=\left(u_1,u_2,0\right)$, $\vec{v}=\left(v_1,v_2,0\right)$, $\overrightarrow{H}=\left(H_1,H_2,0\right)$, *p, ρ, μ, η, σ* are the fluid velocity vector, dust velocity vector, the magnetic field vector, fluid pressure, fluid density, magnetic permittivity, the kinematic coefficient of viscosity and the electric conductivity respectively. Also *m* is the mass of each dust particle, *N* the number density of dust particles and *K* the Stoke's resistance coefficient for the particles. Moreover $\vec{J}=curl\,\overrightarrow{H}$ is the electric field vector.

The velocities of fluid and dust particles are parallel everywhere as, for some scalar α

$$\vec{v}=\frac{\alpha}{N}\vec{u}\,, \tag{2.9}$$

where $\vec{u}\cdot grad\,\alpha=0$. (2.10)

O.P. Chandna et.al. [14], showed that for $q\left(x,y\right)\neq 0$, equations (2.4) and (2.5) are equivalent to a single equation

$$grad\left(\ln q\right)=-\frac{\sigma}{\varepsilon}\frac{\vec{u}}{\vec{u}.\,\vec{u}}+\frac{\left(curl\,\vec{u}\right)\times\vec{u}}{\vec{u}.\,\vec{u}} \tag{2.11}$$

Vorticity function, current density function and the Bernoulli function are defined as

$$\xi=\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y}, \tag{2.12}$$

$$\Omega=\frac{\partial H_2}{\partial x}-\frac{\partial H_1}{\partial y}, \tag{2.13}$$

and $B=p+\frac{1}{2}\rho u^2$ (2.14)

respectively, where $u^2=u_1^2+u_2^2$.

From (2.1), we get

$$\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}=0 \tag{2.15}$$

By using (2.12), (2.13) and (2.14) in (2.6), we get

$$\eta\frac{\partial\xi}{\partial y}-\rho\xi u_2+\mu\Omega H_2-K\left(\alpha-N\right)u_1+\frac{q^2}{\sigma}u_1=-\frac{\partial B}{\partial x}a \tag{2.16}$$

$$\eta\frac{\partial\xi}{\partial x}-\rho\xi u_1+\mu\Omega H_1+K\left(\alpha-N\right)u_2-\frac{q^2}{\sigma}u_2=\frac{\partial B}{\partial y} \tag{2.17}$$

By virtue of (2.4) and (2.5), (2.11) gives

$$\frac{\partial}{\partial x}\left(\ln q\right)+\frac{\sigma}{\varepsilon}\frac{u_1}{u_1^2+u_2^2}+\frac{u_2\xi}{u_1^2+u_2^2}=0 \tag{2.18}$$

$$\frac{\partial}{\partial y}\left(\ln q\right)+\frac{\sigma}{\varepsilon}\frac{u_2}{u_1^2+u_2^2}-\frac{u_1\xi}{u_1^2+u_2^2}=0\,. \tag{2.19}$$

Expressing (2.3) in terms of components, we get

$$u_1H_2-u_2H_1=f\,, \tag{2.20}$$

where *f* is an arbitrary constant.

Similarly, expressing (2.8) in terms of components, we get

$$\frac{m\alpha}{N}\left(\frac{\alpha}{N}\left(u_1\frac{\partial u_2}{\partial x}+u_2\frac{\partial u_2}{\partial y}\right)+u_1\left(u_1\frac{\partial}{\partial x}\left(\frac{\alpha}{N}\right)+u_2\frac{\partial}{\partial y}\left(\frac{\alpha}{N}\right)\right)\right)=K\left(\frac{\alpha}{N}-1\right)u_1 \tag{2.21}$$

$$\frac{m\alpha}{N}\left(\frac{\alpha}{N}\left(u_1\frac{\partial u_2}{\partial x}+u_2\frac{\partial u_2}{\partial y}\right)+u_2\left(u_1\frac{\partial}{\partial x}\left(\frac{\alpha}{N}\right)+u_2\frac{\partial}{\partial y}\left(\frac{\alpha}{N}\right)\right)\right)=K\left(\frac{\alpha}{N}-1\right)u_2 \tag{2.22}$$

Likewise, from (2.2), we get

$$\frac{\partial H_1}{\partial x}+\frac{\partial H_2}{\partial y}=0. \tag{2.23}$$

Now, for a constantly inclined plane flow, let β be the constant non – zero angle between $\vec{u}$ and $\vec{H}$. Then

$$\begin{aligned}u_1H_2-u_2H_1&=uH\sin\beta=f\\ u_1H_1+u_2H_2&=uH\cos\beta=f\cot\beta\end{aligned} \tag{2.24}$$

Solving (2.22), we get

$$H_1=\frac{f}{u^2}\left(cu_1-u_2\right),\; H_2=\frac{f}{u^2}\left(cu_2+u_1\right), \tag{2.25}$$

where $c=\cot\beta$.

Substituting (2.23) in (2.16) and (2.17), we get

$$\eta\frac{\partial\xi}{\partial y}-\rho\xi u_2+\mu\Omega\frac{f}{u^2}\left(cu_2+u_1\right)-K\left(\alpha-N\right)u_1+\frac{q^2}{\sigma}u_1=-\frac{\partial B}{\partial x} \quad (2.26)$$

$$\eta\frac{\partial\xi}{\partial x}-\rho\xi u_1+\mu\Omega\frac{f}{u^2}\left(cu_1-u_2\right)+K\left(\alpha-N\right)u_2-\frac{q^2}{\sigma}u_2=\frac{\partial B}{\partial y} \quad (2.27)$$

respectively.

Further, (2.21) becomes

$$\left(u_2^2-u_1^2-2cu_1u_2\right)\left(\frac{\partial u_1}{\partial y}+\frac{\partial u_2}{\partial x}\right)+\left(cu_2^2-cu_1^2+2u_1u_2\right)\left(\frac{\partial u_1}{\partial x}-\frac{\partial u_2}{\partial y}\right)=0 \quad (2.28)$$

and (2.13) becomes

$$\frac{\partial}{\partial x}\left[\frac{f}{u^2}\left(cu_2+u_1\right)\right]-\frac{\partial}{\partial y}\left[\frac{f}{u^2}\left(cu_1-u_2\right)\right]=\frac{\Omega}{f}. \quad (2.29)$$

The unknown functions to be determined are now u_1, u_2, ξ, Ω, B, q and N. This will lead us to the study of flows.

Let the flow variables $u_1(x, y)$, $u_2(x, y)$ be such that, in the flow region under consideration, the Jacobian $J=\frac{\partial\left(u_1,u_2\right)}{\partial\left(x,y\right)}$ satisfies $0 < J < \infty$.

In such a case, we apply hodograph transformations, where x and y are considered as functions of u_1 and u_2 such that the following conditions hold true:

$$\frac{\partial u_1}{\partial x}=J\frac{\partial y}{\partial u_2}, \qquad \frac{\partial u_1}{\partial y}=-J\frac{\partial x}{\partial u_2},$$
$$\frac{\partial u_2}{\partial x}=-J\frac{\partial y}{\partial u_1}, \qquad \frac{\partial u_2}{\partial y}=J\frac{\partial x}{\partial u_1} \quad (2.30)$$

Using (2.30) in (2.15) and (2.28), we get

$$\frac{\partial x}{\partial u_1}+\frac{\partial y}{\partial u_2}=0 \quad (2.31)$$

$$\left(u_1^2-u_2^2+2cu_1u_2\right)\left(\frac{\partial x}{\partial u_2}+\frac{\partial y}{\partial u_1}\right)+\left(cu_1^2-cu_2^2-2u_1u_2\right)\left(\frac{\partial x}{\partial u_1}-\frac{\partial y}{\partial u_2}\right)=0 \quad (2.32)$$

respectively.

The equation of continuity implies the existence of stream function ψ (x, y) so that

$$\frac{\partial\psi}{\partial x}=-u_2, \quad \frac{\partial\psi}{\partial y}=u_1. \quad (2.33)$$

Likewise equation (2.25) implies the existence of Legendre transform function L (u_1, u_2), so that

$$\frac{\partial L}{\partial u_1}=-y, \quad \frac{\partial L}{\partial u_2}=x. \quad (2.34)$$

Using (2.27) in (2.25), and introducing the polar coordinates (u, θ) in the hodograph plane (u_1, u_2) with $u_1=u\cos\theta,\ u_2=u\sin\theta$, we get

$$\frac{\partial^2 L}{\partial\theta^2}+2cu\frac{\partial^2 L}{\partial u\partial\theta}-u^2\frac{\partial^2 L}{\partial u^2}-2c\frac{\partial L}{\partial\theta}+u\frac{\partial L}{\partial u}=0, \quad (2.35)$$

so that $u=\sqrt{u_1^2+u_2^2}$ and $\theta=\tan^{-1}\left(\frac{u_2}{u_1}\right)$.

III. VORTEX FLOW

A solution of (2.35) is given by

$$L=B_2u^2+\left(A_1\cos\theta+B_1\sin\theta\right)u=B_2\left(u_1^2+u_2^2\right)+A_1u_1+B_1u_2 \quad (3.1)$$

where A_1, B_1 and B_2 are arbitrary constants and $B_2\neq 0$.

Now, using (3.1) in (2.34), we get

$$x=\frac{\partial L}{\partial u_2}=2B_2u_2+B_1,\ y=\frac{\partial L}{\partial u_1}=-\left(2B_2u_1+A_1\right) \quad (3.2)$$

and therefore the fluid velocity field is given by

$$u_1=-\frac{y+A_1}{2B_2}, \qquad u_2=\frac{x-B_1}{2B_2} \quad (3.3)$$

The above relation represents a circulatory flow. Substituting, (3.3) in (2.25), we get

$$H_1=\frac{-2B_2f\left[\left(x-B_1\right)+c\left(y+A_1\right)\right]}{\left(x-B_1\right)^2+\left(y+A_1\right)^2}$$
$$H_2=\frac{2B_2f\left[c\left(x-B_1\right)-\left(y+A_1\right)\right]}{\left(x-B_1\right)^2+\left(y+A_1\right)^2} \quad (3.4)$$

Hence by virtue of (2.12), (2.13), the vorticity function ξ and the current density Ω are respectively given by

$$\xi=\frac{1}{B_2} \quad \text{and} \quad \Omega=0 \quad (3.5)$$

Using (3.3), (3.5) in (2.16), (2.17) and by virtue of the integrability condition for B, we obtain

$$K\left[\left(x-B_1\right)\frac{\partial}{\partial x}\left(\alpha-N\right)+\left(y+A_1\right)\frac{\partial}{\partial y}\left(\alpha-N\right)\right]+$$
$$2K\left(\alpha-N\right)-\frac{2q}{\sigma}\left[\left(x-B_1\right)\frac{\partial q}{\partial x}+\left(y+A_1\right)\frac{\partial q}{\partial y}\right]-\frac{2q^2}{\sigma}=0 \quad (3.6)$$

Also, by using (2.4), (2.5), (3.3) equation (3.6) reduces to

$$K\left[\left(x-B_1\right)\frac{\partial}{\partial x}\left(\alpha-N\right)+\left(y+A_1\right)\frac{\partial}{\partial y}\left(\alpha-N\right)\right]$$
$$+2K\left(\alpha-N\right)+\frac{2q^2}{\sigma}=0 \quad (3.7)$$

For the fluid with infinite electric conductivity, equation (3.7) becomes

$$\left[\left(x-B_1\right)\frac{\partial}{\partial x}\left(\alpha-N\right)+\left(y+A_1\right)\frac{\partial}{\partial y}\left(\alpha-N\right)\right]$$
$$+2\left(\alpha-N\right)=0 \quad (3.8)$$

In particular, the approximation of infinite electrical conductivity is not believed to be appropriate for most problems in the field called ‘magneto-aerodynamics’ [15], it should be appropriate in other situations which involve greater ‘magnetic Reynolds numbers’. Such situations may include flows involving high gas temperatures, or flows of liquid metals.

Solving (3.8), the number density of the dust particles $N(x, y)$ is given by

$N=\dfrac{C_1}{\left(x-B_1\right)\left(y+A_1\right)}+\alpha$, where C_1 is an arbitrary constant. Using equations (2.10) and (3.3), we get

$\alpha=C_2\left[\left(x-B_1\right)^2+\left(y+A_1\right)^2\right]$, where C_2 is another arbitrary constant.

Therefore

$$N=\frac{C_1}{\left(x-B_1\right)\left(y+A_1\right)}+C_2\left[\left(x-B_1\right)^2+\left(y+A_1\right)^2\right] \quad (3.9)$$

Using equations (2.16), (2.17) and (3.3), (3.4), (3.5) we get

$$B=\frac{\rho}{4B_2^2}\left[\left(x-B_1\right)^2+\left(y+A_1\right)^2\right]+\frac{KC_1}{2B_2}\log\left(\frac{x-B_1}{y+A_1}\right)+C_3$$

Where C_3 is an arbitrary constant.

Finally the pressure is given by

$$P=C_3+\frac{\rho}{8B_2^2}\left[\left(x-B_1\right)^2+\left(y+A_1\right)^2\right]+\frac{KC_1}{2B_2}\log\left(\frac{x-B_1}{y+A_1}\right) \quad (3.10)$$

In this case the streamlines of the fluid flow are given by

$\left(x-B_1\right)^2+\left(y+A_1\right)^2=\text{constant}$

This shows that the streamlines are concentric circles.

IV. CONCLUSION

If the dust particle is everywhere parallel to the fluid velocity in the steady, plane, constantly inclined EMFD flow of a viscous, incompressible two-phase fluid, then the streamlines are concentric circles and the dust particle number density is given by (3.9). Also the velocity, the magnetic field, the vorticity, the current density and the pressure are given by (3.3), (3.4), (3.5) and (3.10) respectively.

Although the mathematical complexity in solving equations governing by electrical conduction restrict us in making certain assumptions, but further explorations can be continued with the approach of this paper considering different cases of inclination as well other types of EMFD flows. Even it is believed that some problems based on super conductivity can be taken up and various situations can be derived for further analysis.

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