SOLUTIONS OF GAUSS'S HYPERGEOMETRIC EQUATION, LEGUERRE'S EQUATION BY DIFFERENTIAL TRANSFORM METHOD

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Abstract
In this paper, we find the solution of Gauss's hyper geometric equation, Leguerre's equation Using differential transform method (DTM). The solution obtained by DTM converges the exact solution. The results glaring, devotion, flexibility, accurate and is to easy apply.
Keywords: Gauss'shyper geometric equation, Leguerre's equation, Differential Transform Method (DTM)

1. INTRODUCTION AND PRELIMINARIES

The concept of differential transformation method was first introduced by Zhou [1] in 1986. Its main application was to solve linear and non-linear initial value problems in electric circuit analysis. Differential Transform method is an effective and powerful numerical technique that uses Taylor series method for solution of differential equation in the polynomial form. The DTM is used to evaluate the approximation solution by the finite Taylor series and by an iteration procedure described by the transformed equations obtained from the original equation using the operations of differential transformation. Since the main advantage of this method is that it can be applied directly to nonlinear ordinary and partial differential equations without requiring linearization, discretization or perturbation and also it is able to limit the size of computational work while still accurately providing the series solution with fast convergence rate. It has been studied and applied during the last decades widely. DTM has been used to obtain numerical and analytical solutions of ordinary differential equations [4], class of nonlinear intgro-differential with derivative kernel [2], Delay Differential equations [3], system of differential equations [5] ,nonlinear differential equations [6], Volterra integral equations with separable kernels [7], Volterra integral and Integro differential equations with proportional delay [8], Riccati equation with variable coefficient [9],Quadratic Riccati Differential Equation [10],higher order nonlinear Volterra –Fredhalm integro-differential equations [11] , boundary value problems for integro-differential equations [12], Solutions of integral and integro-differential equations systems [13],integral equations and so on.

The following definitions and results are well known

II. DIFFERENTIAL TRANSFORM METHOD

The transformation of the k th derivative of a function z(x) in one variable is defined as follows

$Z(k)=\frac{1}{k!}\left[\frac{d^k z(x)}{dx^k}\right]_{x=0}$

and the inverse differential transform of Z(k) is defined as $z(x) = \sum_{k=0}^{\infty} Y(k)x^k$

the main theorems of the one-dimensional differential transform are

Theorem 1: If $z(x) = p(x)\pm q(x)$, then $Z(k) = P(k)\pm Q(k)$

Theorem 2: If $z(x) = \alpha p(x)$,then $Z(k) = \alpha P(k)$

Theorem 3:If $z(x) = \frac{dp(x)}{dx}$, then $Z(k)=(k+1)P(k+1)$

Theorem 4: If z(x) = $\frac{d^n p(x)}{dx^n}$, then Z(k) = $\frac{(k+n)!}{k!}$P(k+n)
Theorem 5: If z(x) = p(x)q(x), then Z(k) = $\sum_{r=0}^{k} P(r)Q(k-r)$
Theorem 6: If z(x) = x^l ,then Z(k) = $\delta(k-l) = \begin{cases} 1, k = l \\ 0, k \neq l \end{cases}$, where l is integer
Theorem 7: If z(x) = $\int_{x_0}^{x} p(t)dt$, then Z(k) = $\frac{P(k-1)}{k}$,k≥ 1 ,Z(0) =0
Theorem 8: If z (x) = e^{lx} ,then Z(k) = $\frac{l^k}{k!}$, where l is constant
Theorem 9: If z(x) sin(px+l) , then Z(k)= $\frac{p^k}{k!}$sin($\frac{\pi k}{2} + l$), where p, l are constants
Theorem 10: If z(x) = cos(px+l) , then Z(k)= $\frac{p^k}{k!}$cos($\frac{\pi k}{2} + l$), where p, l are constants
Theorem 11: If z(x) = p_1(x)p_2(x)p_3(x)…p_n(x), then Z(k) =
$\sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{k_{n-3}=0}^{k_{n-2}} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} P_1(k_1)P_2(k_2\text{-}k_1)\dots P_{n-2}(k_{n-2}-k_{n-3})P_{n-1}(k_{n-1}-k_{n-2})P_n(k-k_{n-1})$

III. GAUSS'S HYPERGEOMETRIC EQUATION

Many problems of physical interest are described by ordinary or partial differential Equations with appropriate initial or boundary conditions, these problems are us
The equation of the form x (1-x)y^{11}+$\{\gamma-(\alpha+\beta+1)x\}y^1$- $\alpha\beta$y =0 ------(1.1) is called hyper geometric equation where α,β,γ are constants
Let $\gamma \neq 0,-1,-2,...$ then the solution of (1) is given by
${}_2F_1(\alpha,\beta;\gamma,x)$ = 1+$\frac{\alpha.\beta}{1.\gamma}$x+$\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}$$x^2$+ . . . ---------------(1.2) where ${}_2F_1(\alpha,\beta;\gamma,x)$ is called hyperometric function .
The hypergeometric function F($\alpha,\beta;\gamma,x$) is defined only if (i) α and $\beta are real numbers (ii) \gamma$ is any real number such that Let $\gamma \neq 0,-1,-2,...$ (iii) the variable x satisfies $|x|$<1
If either $\alpha or \beta$is negative integer, then F($\alpha,\beta;\gamma,x$) reduces to a polynomial .
Because after finite number of terms, the coefficient of each term will be zero

Solution of Gauss's hypergeometric equation by differential transforms method

Consider the Gauss's hyper geometric equation x(1-x)y^{11}+$\{\gamma-(\alpha+\beta+1)x\}y^1$- $\alpha\beta$y = 0 ----------(3.1)
Apply the differential transform to (1.3) , we have
$\sum_{r=o}^{k} \delta(r-1)\frac{(k+2-r)!}{k!}$Y(k+r-2) -$\sum_{r=o}^{k} \delta(r-2)\frac{(k+2-r)!}{k!}$Y(k+r-2) +$\gamma$(k+1)Y(k+1)-($\alpha$ + β+1)$\sum_{r=0}^{k} \delta(r-1)(k+1-r)Y(k+1-r) - \alpha\beta Y(k) = 0$ ------(1.4)
For k =0, from (1.4), Y(1) = $\frac{\alpha\beta}{\gamma}$
For k=1, from (1.4), Y(2) = $\frac{\alpha(\alpha+1)\beta(\beta+1)}{2.\gamma(\gamma+1)}$,
And so on
The solution of (1.3) is y(x) = Y(0)x^0+Y(1)x^1+Y(2)x^2+. . .
y(x) = 1+$\frac{\alpha.\beta}{1.\gamma}$x+$\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}$$x^2$+ . . . ---------- (3.2)

IV.NUMERICAL EXAMPLES

Example 1

Consider the Gauss's hyper geometric equation
x(1-x)y^{11}+$\{a-(a+2)x\}y^1$- ay =0
here for α =a,$\beta = 1, \gamma$ = a
Solution of example (1) from (3.2) is y(x) = Y(0)x^0+Y(1)x^1+Y(2)x^2+. . .

y(x) = 1+$\frac{\alpha.\beta}{1.\gamma}$x+$\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}$$x^2$+ . . .
y(x) = 1+$\frac{a.1}{1.a}$x+$\frac{a(a+1)1(1+1)}{1.2.a(a+1)}$$x^2$+ . . .
y(x) = 1+x+x^2+ …

which converges to the exact solution $_2F_1(a, 1; a, x) = (1-x)^{-1}$ (i.e for $\alpha = a, \beta = 1, \gamma = a$)

Table-1

x	DTM	EXACT	Error
0	1.00000	1.00000	0.00000
0.1	1.11000	1.11000	0.00000
0.2	1.24000	1.24000	0.00000
0.3	1.39000	1.39000	0.00000
0.4	1.56000	1.56000	0.00000
0.5	1.75000	1.75000	0.00000
0.6	1.96000	1.96000	0.00000
0.7	2.19000	2.19000	0.00000
0.8	2.44000	2.44000	0.00000
0.9	2.71000	2.71000	0.00000
1	3.00000	3.00000	0.00000

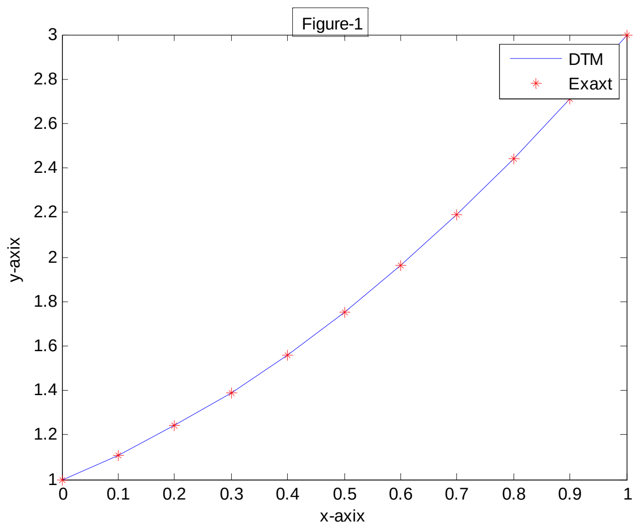


Figure-1

Example 2

Consider the Gauss's hyper geometric equation

$x(1-x)y^{11} + \left\{\frac{1}{2} - (a+b+1)x\right\} y^{1} - aby = 0$

here $\alpha = a, \beta = b, \gamma = \frac{1}{2}$ with $x = \frac{x^2}{4ab}$

Solution of example (2) from (3.2) is $y(x) = Y(0)x^0 + Y(1)x^1 + Y(2)x^2 + \ldots$

$$y(x) = 1 + \frac{\alpha.\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2 + \ldots$$

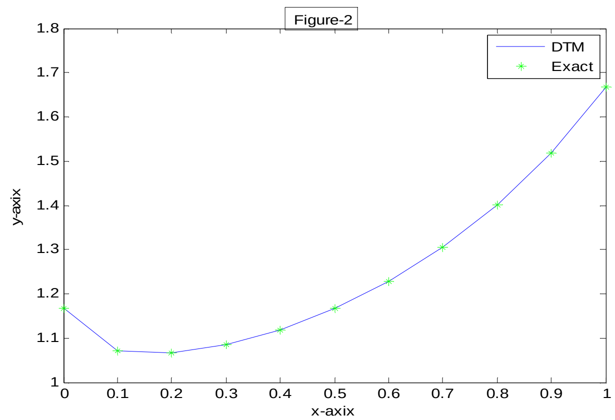
$$y(x) = 1 + \frac{a.b}{1.\frac{1}{2}}\frac{x^2}{4ab} + \frac{a(a+1)b(b+1)}{1.2.\frac{1}{2}(\frac{1}{2}+1)}\frac{x^4}{16a^2b^2} + \ldots$$

$$y(x) = 1 + \frac{x^2}{2!} + (a+1)(b+1)\frac{x^2}{4!} + \ldots$$

which converges to the exact solution $_2F_1(a, b; \frac{1}{2}, \frac{x^2}{ab}) = \cosh x$ (i.e. for $\alpha = a, \beta = b, \gamma = \frac{1}{2}$)

Table-2

x	DTM	EXACT	Error
0	1.166667	1.166667	0.000000
0.1	1.071351	1.071351	0.000000
0.2	1.065991	1.065991	0.000000
0.3	1.084300	1.084300	0.000000
0.4	1.118472	1.118472	0.000000
0.5	1.166667	1.166667	0.000000
0.6	1.228912	1.228912	0.000000
0.7	1.306394	1.306394	0.000000
0.8	1.401609	1.401609	0.000000
0.9	1.519057	1.519057	0.000000
1	1.666667	1.666667	0.000000



Example 3

Consider the Gauss's hyper geometric equation

$x(1\text{-}x)y^{11}+\left\{\frac{3}{2}-2x\right\}y^{1}+2y=0$

here $\alpha=2, \beta=-1, \gamma=\frac{3}{2}$

Solution of example (3) from (3.2) is $y(x) = Y(0)x^0+Y(1)x^1+Y(2)x^2+\ldots$

$$y(x) = 1+\frac{\alpha.\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\ldots$$

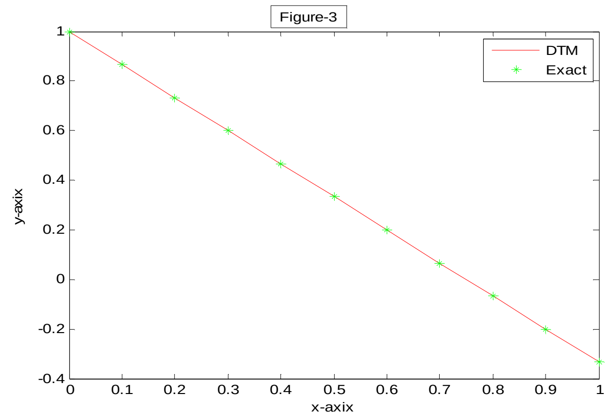
$$y(x) = 1+\frac{2.(-1)}{1.\frac{3}{2}}x+\frac{2(2+1)(-1)(-1+1)}{1.2.\frac{3}{2}(\frac{3}{2}+1)}x^2+\ldots$$

$$y(x) = 1-\frac{4}{3}x+\cdots$$

which converges to the exact solution $_2F_1(2,-1;\frac{3}{2},x)=1-\frac{4}{3}x$ (i.e for $\alpha=2, \beta=-1, \gamma=\frac{3}{2}$)

Table-3

x	DTM	EXACT	Error
0	1.00000	1.00000	0.00000
0.1	0.86666	0.86666	0.00000
0.2	0.73333	0.73333	0.00000
0.3	0.60001	0.60001	0.00000
0.4	0.46680	0.46680	0.00000
0.5	0.33335	0.33335	0.00000
0.6	0.20002	0.20002	0.00000
0.7	0.06669	0.06669	0.00000
0.8	-0.0666	-0.0666	0.00000
0.9	-1.1999	-1.1999	0.00000
1	-0.3333	-0.3333	0.00000



Example 4

Consider the Gauss's hyper geometric equation

$x(1-x)y^{11}+\{1-(-n+2)x\}y^{1}+ny=0$

here $\alpha=-n,\beta=1,\gamma=1$

Solution of example (4) from (3.2) is $y(x) = Y(0)x^0+Y(1)x^1+Y(2)x^2+\ldots$

$y(x) = 1+\frac{\alpha.\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\ldots$

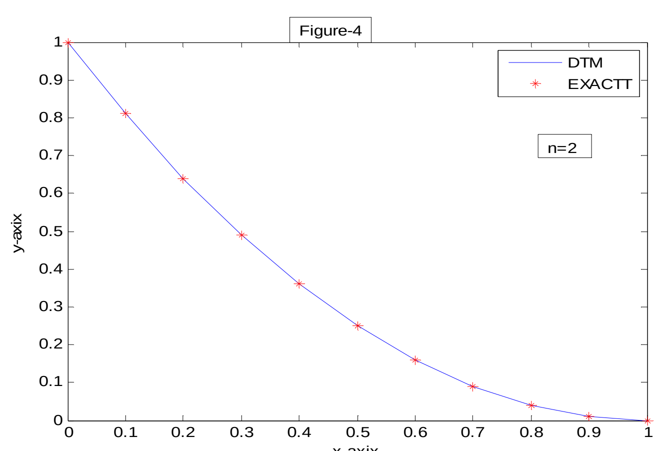
$$y(x) = 1+\frac{-n.1}{1.1}x+\frac{-n(-n+1)(1)(1+1)}{1.2.1(1+1)}x^2+\ldots$$

$y(x) = 1-nx+\frac{n(n-1)}{2!}x^2+\ldots$

which converges to the exact solution $_2F_1(-n, 1; 1, x)=(1+x)^n$ (i.e for $\alpha=-n,\beta=1,\gamma=\frac{3}{2}$)

Table-4

x	DTM	EXACT	Error
0	1.00000	1.00000	0.00000
0.1	0.81000	0.81000	0.00000
0.2	0.64000	0.64000	0.00000
0.3	0.49000	0.49000	0.00000
0.4	0.36000	0.36000	0.00000
0.5	0.25000	0.25000	0.00000
0.6	0.16000	0.16000	0.00000
0.7	0.09000	0.09000	0.00000
0.8	0.04000	0.04000	0.00000
0.9	0.01000	0.01000	0.00000
1	0.00000	0.00000	0.00000



V. SOLUTION OF LEGUERRE'S EQUATION BY DIFFERENTIAL TRANSFORMS METHOD.

The differential equation of the form $xy^{11}+(1-x)y^{1}+n)y= 0$ -------(4.1)
is called Legendre's equation, where n is a positive integer. With initial conditions y(0) = 1,
Apply the Differential Transform Method to equation (4.1)

$$\sum_{r=o}^{k} \delta(r-1)\frac{(k+2-r)!}{k!}Y(k+r-2) + (k+1)Y(k+1) - \sum_{r=o}^{k} \delta(r-1)\frac{(k+1-r)!}{k!}Y(k+1-r) + nY(k) = 0$$

--------- (4.2)

For k = 0, from (4.2), Y(1) = -n

For k= 1, from (4.2), $Y(2) = \frac{n(n-1)}{4}$

For k= 2, from (4.2), $Y(3) = \frac{n(n-1)(n-2)}{36}$

And so on

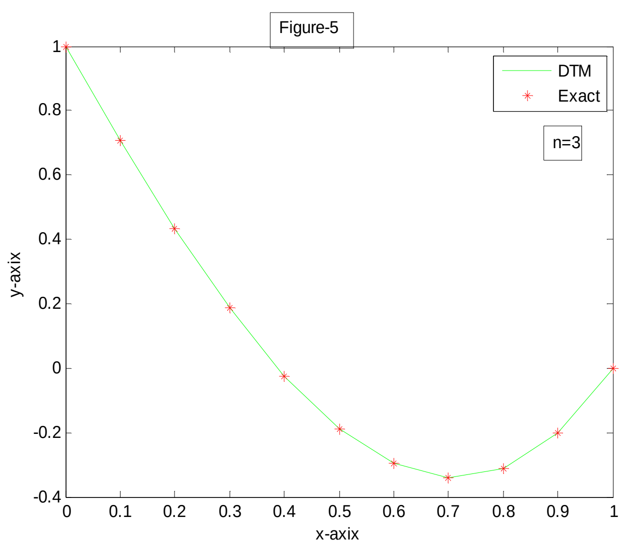
The solution is $y(x) = Y(0)x^0+Y(1)x^1+Y(2)x^2+. . .$

$$y(x) = 1-nx+\frac{n(n-1)}{4}x^2 -- \frac{n(n-1)(n-2)}{36}x^3 + . . .$$

which converges the exact solution of Laguerre 's polynomial of order n $L_n(x) = \sum_{r=0}^{n}(-1)^r \frac{n!}{(n-r)!(r!)^2}x^r$

Table-5

x	DTM	EXACT	Error
0	1.00000	1.00000	0.00000
0.1	0.71500	0.71500	0.00000
0.2	0.46000	0.46000	0.00000
0.3	0.23500	0.23500	0.00000
0.4	0.04000	0.04000	0.00000
0.5	-0.12500	-0.12500	0.00000
0.6	-0.26000	-0.26000	0.00000
0.7	-0.36500	-0.36500	0.00000
0.8	-0.44000	-0.44000	0.00000
0.9	-0.48500	-0.48500	0.00000
1	-0.50000	-0.50000	0.00000



Conclusion

In this paper, Differential Transform Method (DTM) is applied to solve Gauss's hyper geometric equation, Laguerre's equation. we have also given the graphical representation of our findings. The results of DTM and exact solution are in strong agreement with each other. The test problems in this paper shows that the DTM is reliable powerful and very accurate.

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