LOCAL APPROXIMATION BY A CERTAIN FAMILY OF LINEAR POSITIVE OPERATORS

Manoj Kumar
Department of Mathematics and Statistics, Gurukula Kangri Vishwavidyalaya, Haridwar, Uttrakhand, India

Abstract
In 1988, Felten obtained local and global approximation theorems for positive linear operators. In the last decade, Finta studied direct local and global approximation theorems for Baskakov type operators and Szasz-Mirakian type operators. In the present paper, we establish some local approximation estimates for a new family of beta operators, by using Ditzian-Totik modulus of smoothness.

Key Words and Phrases: Local approximation, Linear positive operators, Beta operators, K-functional, Ditzian-Totik modulus of smoothness.

MSC : 41A17, 41A36.

I. Introduction

In 1985, Upreti[10] studied approximation properties of beta operators[4, 5, 8]. Zhou [12] obtained direct and inverse theorems for these operators. Gupta and Dogru [6] obtained some direct results for beta operators. Recently, Kumar [9] studied direct results for Beta-Szasz operators in simultaneous approximation. Motivated by the work on beta operators, Gupta et al. [7] introduced a new family of beta operators to approximate lebesgue integrable functions on $[0,\infty)$ as

$$B_n(f,x)=\frac{1}{(n+1)}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty} b_{n,v}(t)\,f(t)\,dt,\qquad x\in[0,\infty)\tag{1.1}$$

where $n\in N(\text{the set of natural numbers})$, $\quad b_{n,v}(t)=\dfrac{1}{\beta(v,n+1)}t^{v-1}(1+t)^{-n-v-1}$ and

$\beta(v,n+1)=(v-1)!n!/(n+v)!$ the beta function.

Some basic properties of $b_{n,v}(x)$ are as follows

(i). $$\int_0^{\infty} b_{n-r,v+r}(t)\;dt=1\tag{1.2}$$

(ii). $$\int_0^{\infty} t\,b_{n,v}(t)\;dt=\frac{v}{n}\tag{1.3}$$

(iii). $$\int_0^{\infty} t^2\,b_{n,v}(t)\;dt=\frac{v(v+1)}{n(n-1)}\tag{1.4}$$

(iv). $$\sum_{v=1}^{\infty} b_{n,v}(x)=(n+1)\tag{1.5}$$

(v). $$\sum_{v=1}^{\infty} v\,b_{n,v}(x)=(n+1)[1+(n+2)x]\tag{1.6}$$

(vi). $\sum_{v=1}^{\infty} v^2 b_{n,v}(x) = (n+1)[1+3(n+2)x+(n+2)(n+3)x^2]$ (1.7)

(vii). $x(1+x)b'_{n,v}(x) = [(v-1)-(n+2)x]b_{n,v}(x)$ (1.8)

where $n \in N$ and $x \in [0,\infty)$.

It can be easily verified that the operators (1.1) are linear positive operators and the order of approximation by these operators is at best O(n^{-1}) as $n \to \infty$, howsoever smooth the function may be.

Let $£_1^{r}[0,\infty)$ be the class of functions g defined by

$$£_1^{r}[0,\infty) = \{ g : g^{(r)} \in L_1[0,a] \text{ for every } a \in (0,\infty) \text{ and } \left| g^{(r)}(t) \right| \le M(1+t)^m \},$$

where the constants M and m depend on g, and $L_p[a,b], 0 < p < \infty$ stands for the L_p-space.

It is obvious that $L_p^{r}[0,\infty)$ is not contained in $£_1^{r}[0,\infty)$.

Due to Ditzian and Totik[1], the modulus of smoothness of a function f is defined by

$$\omega_{\phi}^2(f,t) = \sup_{0<h\le t} \left\| \Delta_{h\phi}^2 f \right\|_p \tag{1.9}$$

where $\phi^2(x) = x(1+x)$ and

$$\Delta_h^2 f(x) = \begin{cases} f(x-h) - 2f(x) + f(x+h) & \text{if } [x-h, x+h] \subset [0,\infty) \\ 0 & \text{otherwise} \end{cases}$$

Let $\overline{W}_p^2(\phi,[0,\infty)) = \left\{ g \in L_p[0,\infty) : g' \in AC_{loc}[0,\infty) \text{ and } \phi^2 g'' \in L_p[0,\infty) \right\}$.

Following [14], it can be easily verified that the modulus of smoothness [1] defined by (1.9) , is equivalent to the modified K-functional given by

$$\overline{K}_{\phi}^2(f,t^2) = \inf\left\{ \|f - g\|_p + t^2\|\phi^2 g''\|_p + t^4\|g''\|_p : g \in \overline{W}_p^2(\phi,[0,\infty)) \right\}.$$

The main object of the present paper is to establish some local approximation estimates [2, 3] for the operators (1.1) in terms of Ditzian-Totik modulus of second order.

II. Preliminary Results

This section consists of some auxiliary results, which will be helpful in proving the main results of next section.

Lemma 2.1. For $m, r \in N^0$ (the set of non-negative integers), let the function $T_{r,n,m}(x)$ be defined as

$$T_{r,n,m}(x) = \frac{1}{(n+r+1)} \sum_{v=1}^{\infty} b_{n+r,v}(x) \int_0^{\infty} b_{n-r,v+r}(t)(t-x)^m \, dt .$$

Then $T_{r,n,0}(x) = 1$, $T_{r,n,1}(x) = \dfrac{(1+r)(1+2x)}{(n-r)}$ $(n > r)$,

and for all $n > m + r$, there holds the recurrence relation

$$(n-m-r)T_{r,n,m+1}(x) = x(1+x)[T'_{r,n,m}(x) + 2mT_{r,n,m-1}(x)] + (m+r+1)(1+2x)T_{r,n,m}(x).$$

Consequently, for each $x \in [0,\infty)$, we have

$$T_{r,n,m}(x) = O\left(n^{-[(m+1)/2]}\right),$$

where $[\alpha]$ denotes the integral part of α.

Proof. Using the definition of $T_{\mathrm{r},n,m}(x)$ and basic properties of $b_{n,v}(x)$, we obtain

$$T_{\mathrm{r},n,0}(x)=1 \qquad \text{and} \qquad T_{\mathrm{r},n,1}(x)=\frac{(1+r)(1+2x)}{(n-r)}$$

Now, we have

$$x(1+x)T'_{\mathrm{r},n,m}(x)=\frac{x(1+x)}{(n+r+1)}\sum_{v=1}^{\infty}b'_{n+\mathrm{r},v}(x)\int_0^{\infty}b_{n-\mathrm{r},v+\mathrm{r}}(t)\,(t-x)^m\,dt-mx(1+x)T_{\mathrm{r},n,m-1}(x)$$

Therefore, using (1.8) we get

$$x(1+x)[T'_{\mathrm{r},n,m}(x)+mT_{\mathrm{r},n,m-1}(x)]=\frac{1}{(n+r+1)}\sum_{v=1}^{\infty}[(v-1)-(n+r+2)x]\,b_{n+\mathrm{r},v}(x)$$

$$\times\int_0^{\infty}b_{n-\mathrm{r},v+\mathrm{r}}(t)\,(t-x)^m\,dt$$

$$=\frac{1}{(n+r+1)}\sum_{v=1}^{\infty}b_{n+\mathrm{r},v}(x)\int_0^{\infty}\left[\left\{(v+r-1)-(n-r+2)\,t\right\}+(n-r+2)(t-x)+r(1+2x)\right]$$

$$\times\quad b_{n-\mathrm{r},v+\mathrm{r}}(t)\,(t-x)^m\,dt$$

$$=\frac{1}{(n+r+1)}\sum_{v=1}^{\infty}b_{n+\mathrm{r},v}(x)\int_0^{\infty}t(1+t)\,b'_{n-\mathrm{r},v+\mathrm{r}}(t)\,(t-x)^m\,dt$$

$$+(n-r+2)\,T_{\mathrm{r},n,m+1}(x)-r(1+2x)\,T_{\mathrm{r},n,m}(x)$$

$$=\frac{1}{(n+r+1)}\sum_{v=1}^{\infty}b_{n+\mathrm{r},v}(x)\int_0^{\infty}[(t-x)^2+(1+2x)(t-x)+x(1+x)]\,b'_{n-\mathrm{r},v+\mathrm{r}}(t)\,(t-x)^m\,dt$$

$$+(n-r+2)\,T_{\mathrm{r},n,m+1}(x)-r(1+2x)\,T_{\mathrm{r},n,m}(x)$$

$$=\frac{1}{(n+r+1)}\sum_{v=1}^{\infty}b_{n+\mathrm{r},v}(x)\left[\int_0^{\infty}b'_{n-\mathrm{r},v+\mathrm{r}}(t)\,(t-x)^{m+2}\,dt+(1+2x)\int_0^{\infty}b'_{n-\mathrm{r},v+\mathrm{r}}(t)\,(t-x)^{m+1}\,dt\right.$$

$$\left.+\,x(1+x)\int_0^{\infty}b'_{n-\mathrm{r},v+\mathrm{r}}(t)\,(t-x)^m\,dt\right]+(n-r+2)\,T_{\mathrm{r},n,m+1}(x)-r(1+2x)\,T_{\mathrm{r},n,m}(x)$$

$$=-(m+2)\,T_{\mathrm{r},n,m+1}(x)-(m+1)(1+2x)\,T_{\mathrm{r},n,m}(x)-mx(1+x)\,T_{\mathrm{r},n,m-1}(x)$$

$$+(n-r+2)\,T_{\mathrm{r},n,m+1}(x)-r(1+2x)\,T_{\mathrm{r},n,m}(x)$$

Thus, collecting the like terms, we get the desired recurrence relation

$$(n-m-r)\,T_{\mathrm{r},n,m+1}(x)=x(1+x)[T'_{\mathrm{r},n,m}(x)+2mT_{\mathrm{r},n,m-1}(x)]+(m+r+1)(1+2x)\,T_{\mathrm{r},n,m}(x)\,.$$

The other consequence easily follows from the above recurrence relation.

Lemma 2.2. For $f \in L_p^{\mathrm{r}}[0,\infty) \cup £_1^{\mathrm{r}}[0,\infty),\ 1 \le p \le \infty,\ n > r(m+1)$ and $x \in [0,\infty)$, we have

$$B_n^{(\mathrm{r})}(f,x) = \frac{\alpha(n,r)}{(n+1)} \sum_{v=1}^{\infty} b_{n+\mathrm{r},v}(x) \int_0^{\infty} b_{n-\mathrm{r},v+\mathrm{r}}(t) f^{(\mathrm{r})}(t)\,dt \qquad (2.1)$$

where $$\alpha(n,r) = \frac{(n+r)!(N-r)!}{(n!)^2} = \prod_{j=1}^{\mathrm{r}} \frac{(n+j)}{[n-(j-1)]}.$$

Proof. By Leibnitz theorem, with the notation $D \equiv \frac{d}{dx}$, we have

$$B_n^{(\mathrm{r})}(f,x) = \frac{1}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}^{(\mathrm{r})}(x) \int_0^{\infty} b_{n,v}(t) f(t)\,dt$$

$$= \frac{1}{(n+1)} \sum_{i=0}^{\mathrm{r}} \sum_{v=i+1}^{\infty} \frac{(n+v)!}{(v-1)!\,n!} \binom{\mathrm{r}}{i} \left(D^i x^{v-1}\right)\left(D^{\mathrm{r}-i}(1+x)^{-(n+v+1)}\right) \int_0^{\infty} b_{n,v}(t) f(t)\,dt$$

$$= \frac{1}{(n+1} \sum_{v=1}^{\infty} \sum_{i=0}^{\mathrm{r}} (-1)^{\mathrm{r}-i} \binom{\mathrm{r}}{\mathrm{i}} \frac{(n+r)!(n+v+r)!}{n!(v-1)!(n+r)!} \frac{x^{v-1}}{(1+x)^{n+r+v+1}} \int_0^{\infty} b_{n,v+i}(t)\, f(t)\,dt$$

$$= \frac{(n+r)!}{n!(n+1)} \sum_{v=1}^{\infty} b_{n+\mathrm{r},v}(x) \int_0^{\infty} (-1)^{\mathrm{r}} \left[\sum_{i=0}^{\mathrm{r}} (-1)^{\mathrm{i}} \binom{\mathrm{r}}{i} b_{n,v+i}(t) \right] f(t)\,dt \qquad (2.2)$$

Again by Leibnitz theorem, we have

$$b_{n-\mathrm{r},v+\mathrm{r}}^{(\mathrm{r})}(x) = \sum_{i=0}^{\mathrm{r}} \frac{(n+v)!}{(n-r)!(v+r-1)!} \binom{\mathrm{r}}{i} \left(D^{\mathrm{r}-i} x^{v+\mathrm{r}-1}\right)\left(D^i (1+x)^{-(n+v+1)}\right)$$

$$= \sum_{i=0}^{\mathrm{r}} (-1)^{\mathrm{i}} \binom{\mathrm{r}}{i} \frac{n!}{(n-r)!} \frac{(n+v+i)!}{(v+i-1)!\,n!} \frac{x^{v+i-1}}{(1+x)^{n+v+i+1}}$$

$$= \frac{n!}{(n-r)!} \sum_{i=0}^{\mathrm{r}} (-1)^{\mathrm{i}} \binom{\mathrm{r}}{i} b_{n,v+i}(x) \qquad (2.3)$$

Thus, from (2.2) and (2.3), we get

$$B_n^{(\mathrm{r})}(f,x) = \frac{\alpha(n,r)}{(n+1)} \sum_{v=1}^{\infty} b_{n+\mathrm{r},v}(x) \int_0^{\infty} (-1)^{\mathrm{r}} b_{\mathrm{n}-\mathrm{r},v+\mathrm{r}}^{(\mathrm{r})}(t) f(t)\,dt$$

Further integrating by parts *r* times, taking *f (t)* as first function, we get the required result (2.1).

Here, it is important to note that the operators defined in (2.1) by

$$B_n^{(\mathrm{r})}(f,x) = (B_n f)^{(\mathrm{r})} \quad , \qquad f \in L_p^{\mathrm{r}}[0,\infty) \cup £_1^{\mathrm{r}}[0,\infty)$$

are not positive. To make these operators positive, we introduce the operators

$$B_{n,\mathrm{r}} f \equiv D^{\mathrm{r}} B_n I^{\mathrm{r}} f \quad , \qquad f \in L_p[0,\infty) \cup £_1[0,\infty)$$

where *D* and *I* stand for differentiation and integration operators respectively.

Thus, for all $f \in L_p[0,\infty) \cup £_1[0,\infty)$ and $n>r(1+m)$, the operators (2.1) can now be identified as

$$(B_{n,\mathrm{r}} f)(x) = \frac{\alpha(n,r)}{(n+1)} \sum_{v=1}^{\infty} b_{n+\mathrm{r},v}(x) \int_0^{\infty} b_{n-\mathrm{r},v+\mathrm{r}}(t) f(t)\,dt.$$

Obviously, the operators $B_{n,r}$ are positive and the estimation

$$\left\| (B_n f)^{(\mathrm{r})} - f^{(\mathrm{r})} \right\|_p \quad , \quad f \in L_p^{\mathrm{r}}[0,\infty)$$

is equivalent to

$$\left\| B_{n,\mathrm{r}} f - f \right\|_p \quad , \quad f \in L_p[0,\infty).$$

III. Local Approximation Results

In this section, we establish direct local approximation theorems for the operators (1.1). Let $C_B[0,\infty)$ be the space of all real valued continuous and bounded functions *f* on $[0,\infty)$ equipped with the norm $\| f \| = \sup_{x\in[0,\infty)} | f(x) |$.

Also let

$$W_\infty^2 = \{ g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty) \}.$$

Then, for $\delta > 0$, the *K*-functional are defined as

$$K_2(f, \delta) = \inf \{ \| f - g \| + \delta \| g'' \| : g \in W_\infty^2 \}.$$

If $\omega(f, \delta)$ is the usual modulus of continuity of the function $f \in C_B[0,\infty)$ defined by

$$\omega(f, \delta) = \sup_{h\in(0,\delta]} \sup_{x\in[0,\infty)} | f(x+h) - f(x) | ,$$

then $\omega_2(f, \delta^{1/2})$ the second order modulus of smoothness is defined as

$$\omega_2(f, \delta^{1/2}) = \sup_{h\in(0,\delta^{1/2}]} \sup_{x\in[0,\infty)} | f(x+2h) - 2f(x+h) + f(x) | .$$

By Theorem 2.4 [11, p. 177], we can find a positive constant C_1 such that

$$K_2(f, \delta) \le C_1 \omega_2(f, \delta^{1/2}) \tag{3.1}$$

Now we start to study the main results of this section, one of which stated as

Theorem 3.1. Let $f \in C_B[0,\infty)$ and $n \ge 2$. Then for every $x \in [0,\infty)$, there exists an absolute constant $C_2 > 0$ such that

$$| B_n(f, x) - f(x) | \le C_2 \omega_2\left(f, \sqrt{\frac{(x+1)(x+2)}{n}} \right) + \omega\left(f, \frac{1+2x}{n} \right) \tag{3.2}$$

Proof. To derive (3.2), we introduce a new operator

$$\hat{B}_n : C_B[0,\infty) \to C_B[0,\infty)$$

defined by

$$\hat{B}_n(f, x) = B_n(f, x) + f(x) - f\left(\frac{1+(n+2)x}{n} \right) \tag{3.3}$$

Using Lemma 2.1 for *r* =0, we get $\hat{B}_n((t-x), x) = 0$.

For $t, x \in [0,\infty)$, by Taylor's expansion of $g \in W_\infty^2$, we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-y) g''(y)\, dy$$

Consequently, we have

$$\hat{B}_n(g, x) - g(x) = \hat{B}_n\left(\int_x^t (t-y) g''(y)\, dy ,\ x \right)$$

$$= \hat{B}_n\left(\int_x^t (t-y) g''(y)\, dy ,\ x \right) - \int_x^{\frac{1+(n+2)x}{n}} \left(\frac{1+(n+2)x}{n} - y \right) g''(y)\, dy \tag{3.4}$$

Since

$$\left| \int_x^t (t-y) g''(y)\, dy \right| \le (t-x)^2 \| g'' \| \tag{3.5}$$

and

$$\left| \int_x^{\frac{1+(n+2)x}{n}} \left(\frac{1+(n+2)x}{n} - y\right) g''(y)\,dy \right| \leq \left(\frac{1+(n+2)x}{n} - x\right)^2 \| g'' \|$$

$$\leq \left(\frac{1+2x}{n}\right)^2 \| g'' \| \leq 4\left(\frac{1+x}{n}\right)^2 \| g'' \|$$

$$\leq \frac{4(x+1)(x+2)}{n^2} \| g'' \| \tag{3.6}$$

therefore, from (3.4), (3.5) and (3.6), we get

$$\left| \hat{B}_n(g, x) - g(x) \right| \leq B_n\left((t-x)^2,\ x\right) \| g'' \| + \frac{4(x+1)(x+2)}{n^2} \| g'' \|.$$

Thus, using Lemma 2.1 for r =0, we obtain

$$\left| \hat{B}_n(g, x) - g(x) \right| \leq \left[\frac{2(n+5)x^2 + 2(n+5)x + 2}{n(n-1)} + \frac{4(x+1)(x+2)}{n^2} \right] \| g'' \|$$

$$\leq \left[\frac{2(n+5)}{(n-1)} + \frac{4}{n} \right] \frac{(x+1)(x+2)}{n} \| g'' \|$$

$$\leq \frac{16}{n}(x+1)(x+2) \| g'' \| \tag{3.7}$$

Further applying Lemma 2.1 for r =0, we get

$$\left| B_n(f, x) \right| \leq \frac{1}{n+1} \sum_{v=1}^{\infty} b_{n,v}(x) \int_0^{\infty} b_{n,v}(t) \left| f(t) \right| dt \leq \| f \|$$

which implies that B_n is a contraction i.e.

$$\| B_n f \| \leq \| f \| \quad \text{for all } f \in C_B[0,\infty).$$

Now from (3.3), we get

$$\left\| \hat{B}_n f \right\| \leq \| B_n f \| + 2\| f \| \leq 3\| f \| \qquad \text{for all } f \in C_B[0,\infty) \tag{3.8}$$

Hence, in view of (3.3), (3.7) and (3.8), we have

$$\left| B_n(f, x) - f(x) \right| \leq \left| \hat{B}_n(f, x) - f(x) \right| + \left| f(x) - f\left(\frac{1+(n+2)x}{n}\right) \right|$$

$$\leq \left| \hat{B}_n(f-g, x) - (f-g)(x) \right| + \left| \hat{B}_n(g, x) - g(x) \right| + \left| f(x) - f\left(\frac{1+(n+2)x}{n}\right) \right|$$

$$\leq 4\| f-g \| + \frac{16}{n}(x+1)(x+2)\| g'' \| + \left| f(x) - f\left(\frac{1+(n+2)x}{n}\right) \right|$$

$$\leq 16\left[\| f-g \| + \frac{(x+1)(x+2)}{n} \| g'' \| \right] + \sup_{t,\, t-[(1+2x)/n] \in [0,\infty)} \left| f\left(t - \frac{(1+2x)}{n}\right) - f(t) \right|$$

$$\leq 16\left[\| f-g \| + \frac{(x+1)(x+2)}{n} \| g'' \| \right] + \omega\left(f,\ \frac{1+2x}{n} \right) \tag{3.9}$$

Finally, taking the infimum on the right hand side of (3.9), over all $g \in W_\infty^2$ and applying (3.1), we get the desired result (3.2).

Our next result of this section is the following

Theorem 3.2. Let $n > r+1 \geq 2$ and $f^{(i)} \in C_B[0,\infty)$, $i = 0(1)r$. Then for every $x \in [0,\infty)$, we have

$$\left| B_n^{(r)}(f,x) - f^{(r)}(x) \right| \leq \left[\frac{(n+r+1)!(n-r)!}{(n+1)!n!} - 1 \right] \| f^{(r)} \|$$

$$+ \frac{(n+r+1)!(n-r)!}{(n+1)!n!} \left[1 + \sqrt{\frac{2\{(n+1)+2(r+1)(r+2)\}x(x+1)+(r+1)(r+2)}{(n-r-1)}} \right]$$

$$\times \ \omega\left(f^{(r)}, (n-r)^{-1/2} \right) \tag{3.10}$$

Proof. Using the basic properties of $b_{n,v}(x)$ and Lemma 2.2, we have

$$B_n^{(r)}(f,x) - f^{(r)}(x) \leq \frac{(n+r)!(n-r)!}{(n+1)!n!} \sum_{v=1}^{\infty} b_{n+r,v}(x) \int_0^\infty b_{n-r,v+r}(t) \left[f^{(r)}(t) - f^{(r)}(x) \right] dt$$

$$+ \left[\frac{(n+r+1)!(n-r)!}{(n+1)!n!} - 1 \right] f^{(r)}(x)$$

Since for every $\delta > 0$, we have

$$\left| f^{(r)}(t) - f^{(r)}(x) \right| \leq \omega\left(f^{(r)}, |t-x| \right) \leq \left(1 + \frac{|t-x|}{\delta} \right) \omega\left(f^{(r)}, \delta \right) ,$$

therefore, we get

$$\left| B_n^{(r)}(f,x) - f^{(r)}(x) \right| \leq \frac{(n+r)!(n-r)!}{(n+1)!n!} \sum_{v=1}^{\infty} b_{n+r,v}(x) \int_0^\infty b_{n-r,v+r}(t) \left| f^{(r)}(t) - f^{(r)}(x) \right| dt$$

$$+ \left[\frac{(n+r+1)!(n-r)!}{(n+1)!n!} - 1 \right] \| f^{(r)} \|$$

$$\leq \frac{(n+r)!(n-r)!}{(n+1)!n!} \sum_{v=1}^{\infty} b_{n+r,v}(x) \int_0^\infty b_{n-r,v+r}(t) \left[1 + \delta^{-1} |t-x| \right] \omega\left(f^{(r)}, \delta \right) dt$$

$$+ \left[\frac{(n+r+1)!(n-r)!}{(n+1)!n!} - 1 \right] \| f^{(r)} \| \tag{3.11}$$

Applying Cauchy-Schwarz inequality for integration and then summation, we obtain

$$\sum_{v=1}^{\infty} b_{n+r,v}(x) \int_0^\infty b_{n-r,v+r}(t) |t-x| dt \leq \sum_{v=1}^{\infty} b_{n+r,v}(x) \left[\int_0^\infty b_{n-r,v+r}(t)\, dt \right]^{1/2} \left[\int_0^\infty b_{n-r,v+r}(t)(t-x)^2\, dt \right]^{1/2}$$

$$\leq \left[\sum_{v=1}^{\infty} b_{n+\mathrm{r},v}(x) \right]^{1/2} \left[\sum_{v=1}^{\infty} b_{n+\mathrm{r},v}(x) \int_0^{\infty} b_{n-\mathrm{r},v+\mathrm{r}}(t)\,(t-x)^2\, dt \right]^{1/2}$$

$$\leq (n+r+1)^{1/2} \left[\sum_{v=1}^{\infty} b_{n+\mathrm{r},v}(x) \int_0^{\infty} b_{n-\mathrm{r},v+\mathrm{r}}(t)\,(t-x)^2\, dt \right]^{1/2} \qquad (3.12)$$

From straight computations, we get

$$\int_0^{\infty} b_{n-\mathrm{r},v+\mathrm{r}}(t)\,(t-x)^2\, dt = \frac{(v+r+1)\,(v+r)}{(n-r)\,(n-r-1)} - 2x\,\frac{(v+r)}{(n-r)} + x^2$$

Thus, we obtain

$$\sum_{v=1}^{\infty} b_{n+\mathrm{r},v}(x) \int_0^{\infty} b_{n-\mathrm{r},v+\mathrm{r}}(t)\,(t-x)^2\, dt$$

$$= \frac{(n+r+1)\left[2\{(n+1)+2(r+1)\,(r+2)\}\,x(x+1)+(r+1)\,(r+2)\right]}{(n-r)\,(n-r-1)} \qquad (3.13)$$

Hence, collecting the estimates of (3.11), (3.12) and (3.13), we get

$$\left| B_n^{(\mathrm{r})}(f,x) - f^{(\mathrm{r})}(x) \right| \leq \left[\frac{(n+r+1)!(n-r)!}{(n+1)!n!} - 1 \right] \left\| f^{(\mathrm{r})} \right\| \; +$$

$$\frac{(n+r+1)!(n-r)!}{(n+1)!n!} \left[1 + \delta^{-1} \sqrt{\frac{2\{(n+1)+2(r+1)(r+2)\}x(x+1)+(r+1)\,(r+2)}{(n-r)\,(n-r-1)}} \right] \omega\left(f^{(\mathrm{r})}, \delta \right)$$

Finally, choosing $\delta = (n-r)^{-1/2}$, we get the required result (4.10).

This completes the proof of the Theorem 3.2.

REFERENCES

[1] Z. Ditzian and V. Totik , *Moduli of smoothness*, Springer Series in Computational Mathematics 9, Springer-Verlag, Berlin, Heidel-berg, New York, 1987.

[2] M. Felten , *Local and Global Approximation Theorems for Positive Linear Operators*, J. Approx. Theory, 94(1988), 396-419.

[3] Z. Finta , *Direct local and global approximation theorems for some linear positive operators*, Analysis in Theory and Applications, 20:4 (2004), 307-322.

[4] V. Gupta, *Stancu variant of Beta-Szasz operators*, Georgian Math. J. 21 (1) (2014), 75-82.

[5] V. Gupta, *Convergence of Szasz Beta operators*, Ann. Univ. Ferrara 59 (2013), 311-318.

[6] V. Gupta and O. Dogru , *Direct results for certain family of integral type operators*, J. Inequal. Pure and Appl. Math., 6(1) Art.21(2005), 1-11.

[7] M. K. Gupta, V. Gupta and M. Kumar, *Direct and inverse estimates for a new family of linear positive operators*, J. Math. Anal. Appl. 330(2007), 799-816.

[8] V. Gupta and R. Yadav, *Some approximation results on q-Beta-Szasz operators*, Southeast Asian Bulletin of Mathematics 36 (3) (2012), 343-352.

[9] M. Kumar, *On simultaneous approximation of functions by a certain family of linear positive operators*, International Journal of Advanced Research in Computer Science and Software Engineering, Vol. 6, Issue 9 (2016), 452-456.

[10] R. Upreti , *Approximation properties of Beta operators*, J. Approx. Theory, 45(1985), 85-89.

[11] R. A. D. Vore and G. G. Lorentz , *Constructive Approximation*, Springer-Verlag, Berlin, (1993).

[12] D. X. Zhou , *L-Inverese theorems for Beta operators*, J. Approx. Theory, 66(1991), 279-287.