MATHEMATICAL MODELLING OF FLOW THROUGH POROUS MEDIA AND DETERMINATION OF VELOCITY POTENTIAL

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Abstract

Seepage is one of the important factors to be considered in proper design of earthen embankments as it pertains to the estimation of quantity of water seeping through the medium in consideration and resulting pore pressure developed due to it. Seepage can occur beneath the foundation and also through the body of the hydraulic structure. Excessive seepage could result in scouring and piping thereby leading to the ultimate failure of the structure. Since analytical solution is quiet difficult to find velocity potential and governing seepage equation, its determination requires an iterative procedure. In the present work, a numerical code in MATLAB software has been developed using FEM to find out the potential head in a flow domain under a sheet pile. The work can be extended to find the velocity potential through anisotropic soil condition.

Index Terms: Seepage, Velocity potential, anisotropic soil.

I. INTRODUCTION

Seepage analysis in an important factor to be considered in the proper design of any civil engineering structures. The term 'seepage' usually refers to situations where the primary driving force is gravity controlled, such as establishing seepage losses from a reservoir, where the driving force is the total hydraulic head difference between the entrance and exit points. Another cause of water movement in soils is the existence of excess pore water pressure due to external loading. This type of water flow is usually not referred to as seepage, but the fundamental mathematical equations describing the water movement are essentially identical. Thus, the term seepage can be used to describe all movement of water through soil regardless of the creation or source of the driving force or whether the flow is through saturated or unsaturated soils. Seepage can occur in both foundations of an engineering structure or sometimes through the structure itself as the case of earthen dams. A failure of earth dam is attributed to the following: hydraulic failure, seepage failure, piping through dam body and structural failure due to earthquake. Seepage analysis pertains to the estimation of quantity of water seeping through the medium in consideration and resulting uplift pressure developed due to it and the exit gradients. Excessive seepage could result in scouring and piping and the ultimate failure of the foundation and the structure accompanying it. The design and construction of an earth-fill dam is one of the key challenges because of the unavoidable variation in foundation condition and the properties of the available construction materials. A homogeneous earth-fill dam should be designed with relatively flat slopes to reduce the risk of failure. The practical seepage problems are not easily convertible into an equivalent numerical counterpart because of the heterogeneity of the natural soils and the varying boundary conditions. Even though the solution of governing equation seems tedious, the solution which is more close to the analytical solution can be derived using finite element formulation. The analysis of seepage can be done by the discharge computation from flownet. A flow net is a graphical solution to the equations of steady groundwater flow. A flow net consists of two sets of lines which must always be orthogonal (perpendicular to each other): flow lines, which show the direction of groundwater flow, and equipotential (lines of constant head), which show the distribution of potential energy. Flow nets are usually constructed through trial-and-error sketching.

II. METHODOLOGY

Modelling the flow of water through soil with a numerical solution can be very complex. Natural soil deposits are generally highly heterogeneous and anisotropic. In addition, boundary conditions often change with time and cannot always be defined with certainty at the beginning of an analysis; in fact, the correct boundary condition can sometimes be part of the solution. When a soil becomes unsaturated, the coefficient of permeability or hydraulic conductivity becomes a function of the negative pore water pressure in the soil. The pore water pressure is the primary unknown and needs to be determined, so iterative numerical techniques are required to match the computed seepage pressure and the material property, which makes the solution highly nonlinear. These complexities make it necessary to use some form of numerical analysis to analyze seepage problems for all, but the simplest cases. A common approach is to use finite element formulations

A. Triangular Element and Shape Function

The division of a two dimensional body into linear triangles is emphasized because this element is analytically the simplest of the two dimensional elements. This simplicity of the element requires that a large number of elements be used to model a region; therefore the division of a domain into linear triangles probably represents the finest subdivision that would be used.

The division of any two dimensional domain into elements should star with the division of body into quadrilateral and triangular regions. These regions are when subdivided into triangles. The subdivision between regions should be located where there is a change in geometry, applied load or material properties or both. A triangular region is most easily divided into elements by specifying the same number of nodes along each side and then connecting the appropriate nodes by straight lines and placing nodes at the intersecting points.

Figure1 shows a typical triangular element having nodes1, 2 and 3

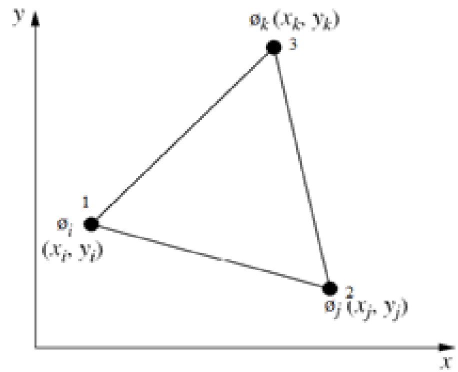

Figure1. Triangular element

At each node, let the value of potential be $\varphi_1, \varphi_2, \varphi_3$ and coordinates (X_i, Y_i), (X_j, Y_j), (X_k, Y_k) respectively. Then the value of potential within the triangular mesh can be expressed as

$$\varphi = N_i\varphi_i + N_j\varphi_j + N_k\varphi_k \quad (1)$$

This can be expressed in matrix form as

$$\begin{bmatrix} N_i \\ N_j \\ N_k \end{bmatrix} [\varphi_i \quad \varphi_j \quad \varphi_k] \quad (2)$$

Where N is called the interpolation function, given by

$$N_i = \frac{1}{2A}(a_i + b_i x + c_i y) \quad (3)$$

$$N_j = \frac{1}{2A}(a_j + b_j x + c_j y) \quad (4)$$

$$N_k = \frac{1}{2A}(a_k + b_k x + c_k y) \quad (5)$$

Where

$$a_i = X_j Y_k - X_k Y_j \quad b_i = Y_j - Y_k \quad c_i = X_k - X_j$$

$$a_j = X_k Y_i - Y_k X_i \quad b_j = Y_k - Y_i \quad c_j = X_i - X_k$$

$$a_k = X_i Y_j - Y_i X_j \quad b_k = Y_i - Y_j \quad c_k = X_j - X_i$$

B. Formulation of finite element equation

The governing equation for steady state flow through porous media

$$\frac{\partial^2\varphi}{\partial x^2}K_x + \frac{\partial^2\varphi}{\partial y^2}K_y = 0 \quad (6)$$

Where K_x and K_y are coefficients of permeability

Considering the weighted residual formulation

$$\iint w\left[\frac{\partial^2\varphi}{\partial x^2}K_x + \frac{\partial^2\varphi}{\partial y^2}K_y\right] dxdy = 0$$

Where w is the weighing function

On integration by parts,

$$\int w \frac{\partial \varphi}{\partial x} + \int w \frac{\partial \varphi}{\partial y} - \int \int \left[\frac{\partial w}{\partial x} \frac{\partial \varphi}{\partial x} K_x + \frac{\partial w}{\partial y} \frac{\partial \varphi}{\partial y} K_y \right] dxdy = 0$$

Since we have $\varphi = \sum N_i \varphi_i$

$$\int Ni \frac{\partial \varphi}{\partial x} + \int Ni \frac{\partial \varphi}{\partial y} - \iint \left[\frac{\partial Ni}{\partial x} \frac{\partial \sum Ni\varphi i}{\partial x} K_x + \frac{\partial Ni}{\partial y} \frac{\partial \sum Ni\varphi i}{\partial y} K_y \right] dxdy = 0$$

This equation can be converted into the general form

$$Q - K\varphi = 0 \quad (7)$$

$$K\varphi = Q \quad (8)$$

Where K is called the stiffness matrix and Q is the boundary condition.

For isotropic soil the stiffness matrix K is given by

$$K = \frac{1}{4A} \begin{bmatrix} b1^2 + a1^2 & b1b2 + a1a2 & b1b3 + a1a3 \\ & b2^2 + a2^2 & b2b3 + a2a3 \\ symmetrical & & b3^2 + a3^2 \end{bmatrix}$$

Where

$a1 = X_k - X_j \quad a2 = X_i - X_k \quad a3 = X_j - X_i$

$b1 = Y_j - Y_k \quad b2 = Y_k - Y_i \quad b3 = Y_i - Y_j$

Where *A* is the area of one triangular mesh

C. Algorithm in FEM

A general program to determine the total potential within an earthen dam was developed in MATLAB software. Algorithm for the determination of velocity potential is written as:

- Discretize into finite elements.
- Identify nodes & elements
- Develop element stiffness matrices [K_e] for all elements
- Assemble element stiffness matrices to get the global stiffness matrix
- Apply boundary conditions
- Solve for velocity potential

With the potential values at each nodes, the equipotential lines were drawn by interpolation method. The stream lines were drawn based on literature survey.

According to Casagrande (Casagrande, 1940), the following rules should be obeyed in drawing flow nets:

- Flow lines and equipotential lines should always be perpendicular to each other, in a homogeneous isotropic system, and form curvilinear “squares”.
- Flow lines should always be parallel to an impermeable boundary, and equipotential lines are always perpendicular to it.
- Flow lines should always be perpendicular to a constant head boundary, and equipotential lines are always parallel to it.

For drawing flow nets the suggested procedure is as follows:

- The boundary conditions should be firstly identified, e.g. boundaries which are impermeable and which are constant head.
- Then, regions where the water is entering the system and regions where it can it should be identified.
- Any symmetry in the boundary conditions should always be looked for.
- The number of flow channels to be used should be decided.
- By trial and error sketching, a first trial flow line must be drawn and then other flow lines must be drawn in to define all flow channels.
- Where flow channels squeeze, more closely spaced equipotential lines (higher head gradients) are required to transfer the same quantity of water through the flow channel.
- Finally, the curvilinear squares should be fitted together by drawing in the equipotential lines. In this process, the positions of certain flow lines may be revised, making a trial and error procedure.

III. NUMERICAL CODE FORMULATION AND ITS VERIFICATION

In order to validate the finite element formulation, we consider an example of flow field with known potential having analytical solution

A. Flow field with known potential

The problem is to find out the velocity potential for a rectangular flow field, when the flow is in the X-direction. The boundary is assumed to be impermeable and there is no flow in Y-direction. The velocities in the X-direction and the field dimensions are known. The boundary conditions are formulated for all the four sides of field. The flow grid is shown in Figure2. Domain for known potential

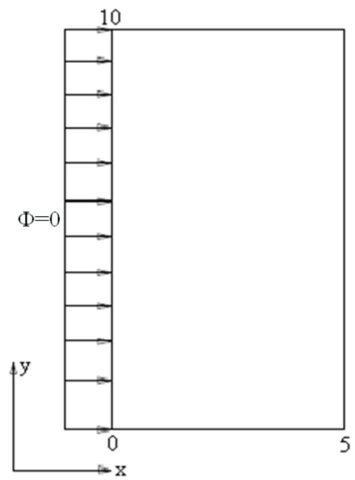

Figure2. Domain for known potential

- The governing equation for the problem is $\nabla^2\phi = 0$
- The flow grid used is a size of 5x10
- Potential distribution=$100 \times \sin\left(\frac{\pi x}{10}\right)$

Analytical solution=$\frac{100 \times \sinh(0.314159265 \times y) \sin(0.314159265 \times x)}{\sinh(3.14159265)}$

B. Discretization

To solve the partial differential equation governing the seepage problem using FEM, first the domain is discretized to elements and nodes and then the approximation function is systematically derived over the given domain. Then, the Galerkin type approximation of the governing equation is derived over each element. Finally, the equations over all elements of the collection are converted by the continuity of primary variables, the boundary conditions of the problem are imposed, and the connection set of equations is solved.

Since we know the upstream and downstream head, the nodes 1, 2, 3, 4, 5, 6, 11, 16, 21, 22, 23, 24 and 25 are considered as constrained nodes.

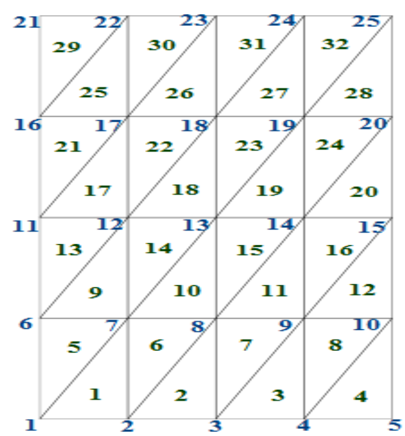

Figure3. Discretization of flow field with known potential

- Number of nodes=25
- Number of elements=32

Since the analytical solution is available for the problem, the developed program '*potential.m*' was used to find the finite element solution of the same problem in MATLAB software. The results from the MATLAB software comprising of both analytical and finite element software is given by

Table1. Comparison of FEM and Analytical solution

Node no.	FEM Solution	Analytical Solution	Absolute Error	Relative Error
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0
4	0	0	0	0
5	0	0	0	0
6	0	0	0	0
7	3.051585	2.882226	0.169359	5.549867
8	5.638593	5.324463	0.31413	5.571079
9	7.367177	6.953889	0.413288	5.609854
10	7.974175	7.521764	0.452411	5.673457
11	0	0	0	0
12	7.961473	7.635645	0.325828	4.092563
13	14.71089	14.10566	0.605222	4.114107
14	19.2207	18.42237	0.798328	4.153482
15	20.80433	19.92679	0.877541	4.218066
16	0	0	0	0
17	17.71961	17.34626	0.373343	2.106948
18	32.74157	32.04452	0.697055	2.128959
19	42.77893	41.85099	0.927941	2.169154
20	46.30358	45.26866	1.034917	2.23507
21	0	0	0	0
22	38.2683	38.31839	0.050093	0.130899
23	70.7107	70.78726	0.076558	0.108269
24	92.388	92.45003	0.062032	0.067143
25	100	99.99977	0.000235	0.000235

From the above table, It is observed that, there is no much variation between analytical solution and FEM solution. Also, it is inferred that the relative error between finite element solution and analytical solution is considerably small and so the program potential.m can be used to find potential value under any flow domain for specified boundary condition.

IV. APPLICATION OF MATLAB PROGRAMME UNDER A FLOW DOMAIN

Consider a flow field of length 36 m, depth 12 m having unit thickness under a hydraulic structure. The upstream water level is 7.5 m and the downstream water level is 1m. The flow field is discretized using Finite Element Method. Figure 5 shows flow domain with a sheet pile of depth 40% of total depth and Figure 6 shows the discretized flow field. The node number 56, 57, 58, 64, 65 and 66 are considered as the constrained nodes.

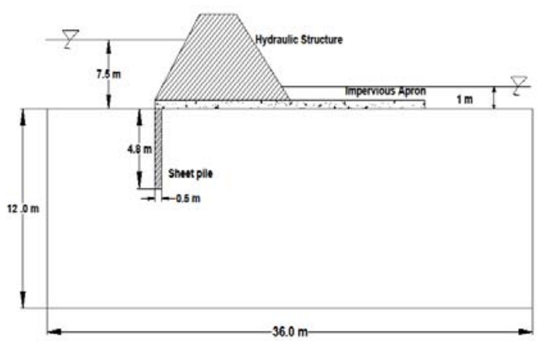

Figure5. Flow domain

Total number of nodes = 66
Governing equation $\nabla^2\phi = 0$
Total number of elements = 94
Flow grid size 36m x 12m

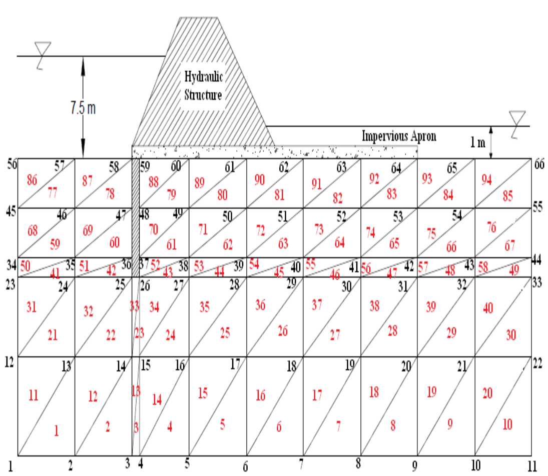

Figure6. Discretized flow domain

NODE no	POTENTIAL	NODE no	POTENTIAL	NODE no	POTENTIAL
1	5.9944584	23	6.5705933	45	7.0851237
2	5.8199957	24	6.4071261	46	7.0182092
3	5.3117132	25	5.7181447	47	6.8380372
4	5.2348195	26	5.3915447	48	4.8156857
5	4.6297403	27	4.5815779	49	4.4552327
6	3.9361162	28	3.8757894	50	3.8332537
7	3.2773441	29	3.1880072	51	3.139658
8	2.6797809	30	2.5007786	52	2.3711327
9	2.1970653	31	1.8695303	53	1.4749889
10	1.9024212	32	1.5770085	54	1.269625
11	1.8055752	33	1.4947507	55	1.2238468
12	6.168921	34	6.7037048	56	7.5
13	5.9869057	35	6.5647327	57	7.5
14	5.4067063	36	6.0859884	58	7.5
15	5.3111888	37	5.1483177	59	4.7184516
16	4.6308629	38	4.545802	60	4.4206721
17	3.9186902	39	3.8632635	61	3.8211482
18	3.2467397	40	3.1732687	62	3.1247796
19	2.6223571	41	2.4647859	63	2.3093841
20	2.1030295	42	1.7772829	64	1
21	1.8035221	43	1.4993536	65	1
22	1.7087292	44	1.4248045	66	1

V. RESULTS AND ANALYSIS

Using the MATLAB program *shhet40.m*, the value of potential under the flow field is determined and tabulated and furnished in following table. With the potential values from the above table, the equipotential lines were drawn by interpolation. Stream lines perpendicular to the equipotential lines were drawn with the guide lines and the flow net was constructed.

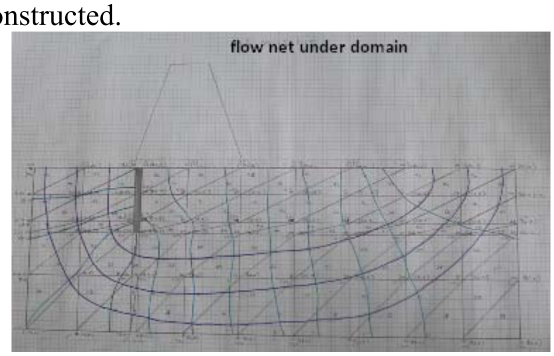

Figure 7. Flow net under flow domain

Since we obtain the flow net under a flow domain, the discharge through the flow field can be calculated with the help of flow net constructed. The discharge through the flow net is given by

$$Q = kH\left(\frac{N_f}{N_d}\right) m^3/s \tag{8}$$

k = Coefficient of permeability = 4.5 x 10^{-6} m/s (Assumed)
H = u/s head – d/s head = 7.5 – 1 = 6.5 m
N_f = Number of flow field = 4
N_d = Number of potential drop= 13
As the flow domain is considered in the two dimensional direction the discharge obtained will be discharge per unit width and discharge,

Q=0.7776m³/day

VI. CONCLUSION

From the present work of determination of potential using finite element method, the following conclusions are derived:

- Mathematical modelling of flow through porous media has been studied.
- A program in MATLAB software has been developed to solve the governing equation of seepage through porous media, which is Laplace equation.
- Using MATLAB Program developed, the velocity potential under a flow domain has been determined under specified boundary conditions.
- Flow net under the flow domain has been constructed and analysed to determine the discharge value.

Discharge Q=0.7776m³/day

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