MATHEMATICAL MODELLING OF TRANSIENT THERMOELASTIC PROBLEM OF A CYLINDER

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ABSTRACT
This paper is concerned with transient thermoelastic problem of a finite length hollow cylinder to determine the temperature gradient, displacement and stress functions at any point of the cylinder. The integral transform techniques are used to find the solution of the problem. The results are expressed in terms of Bessel's function in the form of infinite series and depicted graphically.

Keywords:Transient, Transform, Integral.

INTRODUCTION

Sirakowski and Sun (1968) studied the direct problems of finite length hollow cylinder and determined an exact solution. Grysa and Cialkowski (1980) and Grysa and Kozlowski (1982) discussed one dimensional transient thermoelastic problems derived the heating temperature and the heat flux on the surface of an isotropic infinite slab. Further Deshmukh and Wankhede (1997) studied an axisymmetric inverse steady state problem of thermoelastic deformation to determine the temperature, displacement and stress functions on the outer curved surface of finite length hollow cylinder.

In this paper, an attempt has been made to determine the temperature gradient, displacement and stress functions at any point of the cylinder.

BASIC EQUATIONS

The differential equations of temperature, thermoelastic displacement function of a transient finite length hollow cylinder of length $2h$ in the absence of body forces, heat sources are

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k}\frac{\partial T}{\partial t} \quad (1)$$

$$\nabla^2 \phi = \frac{(1+\nu)}{(1-\nu)} a_t T \quad (2)$$

with $\phi = 0$ at $r = a$ and $r = b$ (3)

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$

ν and a_t are the Poisson's ratio and the linear coefficient of thermal expansion of the material of the cylinder.

FORMULATION AND SOLUTION OF THE PROBLEM

Consider a hollow cylinder of length $2h$ occupying space D defined by $a \le r \le b$; The initial temperature in the thick plate is given by a constant temperature T_0, and the heat flux $f(r,t)$ is prescribed on the upper and lower boundary surfaces. Under these conditions, the thermoelastic quantities in a hollow cylinder are required to be determined. We take a cylindrical polar co-ordinate system with symmetry about z –axis. The radial and axial displacements U and W satisfying the uncoupled thermoelastic equations are

$$\nabla^2 U - \frac{U}{r^2} + (1+2\nu)^{-1} \frac{\partial e}{\partial r} = 2\frac{(1+\nu)}{(1-2\nu)} a_t \frac{\partial T}{\partial r} \tag{4}$$

$$\nabla^2 W + (1+2\nu)^{-1} \frac{\partial e}{\partial z} = 2\frac{(1+\nu)}{(1-2\nu)} a_t \frac{\partial T}{\partial z} \tag{5}$$

Where $e = \frac{\partial U}{\partial r} + \frac{U}{r} + \frac{\partial W}{\partial z}$ is the volume dilatation and

$$U = \frac{\partial \phi}{\partial r} \tag{6}$$

$$W = \frac{\partial \phi}{\partial z} \tag{7}$$

and Constitutive relations

$$\tau_{rz}(a, z, t) = 0, \quad \tau_{rz}(b, z, t) = 0, \quad \tau_{rz}(r, z, 0) = 0, \tag{8}$$

$$\sigma_r(a, z, t) = p_1, \quad \sigma_r(b, z, t) = -p_0, \quad \sigma_z(r, 0, t) = 0, \tag{9}$$

Where p_1 and p_0 are the surface pressures assumed to be uniform over the boundaries of the cylinder. The boundary conditions for the stress functions (8) and (9) are expressed in terms of the displacement components by the following relations:

$$\sigma_r = (\lambda + 2G)\frac{\partial U}{\partial r} + \lambda\left[\frac{U}{r} + \frac{\partial W}{\partial z}\right] \tag{10}$$

$$\sigma_z = (\lambda + 2G)\frac{\partial W}{\partial z} + \lambda\left[\frac{\partial U}{\partial r} + \frac{U}{r}\right] \tag{11}$$

$$\sigma_\theta = (\lambda + 2G)\frac{U}{r} + \lambda\left[\frac{\partial U}{\partial r} + \frac{\partial W}{\partial z}\right] \tag{12}$$

$$\tau_{rz} = G\left[\frac{\partial W}{\partial r} + \frac{\partial U}{\partial z}\right] \tag{13}$$

where $\lambda = \frac{2G\nu}{1-2\nu}$ is the Lame's constant, G is the shear modulus and U and W are the displacement components.

Applying transform defined in Appendix A to the equation (1), one obtains

$$-\mu_m^2 \overline{T} + \frac{d^2\overline{T}}{dz^2} = \frac{1}{k}\frac{d\overline{T}}{dt} \tag{14}$$

Further applying transform in Appendix B to the equation (14), one obtains

$$\frac{d\overline{T}^*}{dt} + kp^2\overline{T}^* = k\omega(m,t) \tag{15}$$

where $p^2 = a_n^2 + \mu_m^2$

$$\omega(m,t) = \frac{P_n(h)}{\alpha_1}\bar{f}_2(m,t) - \frac{P_n(-h)}{\alpha_2}\bar{f}_1(m,t)$$

Equation (15) is the first order differential equation, whose solution is given by

$$\overline{T}^* = e^{-kp^2 t}[\overline{F}^* + X + k\int_0^t \omega(m,t')e^{kp^2 t'}dt'] \tag{16}$$

Substituting the value of $T(r,z,t)$ from equation (16) in equation (2), one obtains the thermoelastic displacement function $\phi(r,z,t)$ as

$$\phi(r,z,t)=\frac{a_t r^2}{4}\frac{(1+\nu)}{(1-\nu)}\sum_{m,n=1}^{\infty}\frac{S_0(k_1,k_2,\mu_m r)P_n(z)}{\mu_m\lambda_n} \times e^{-kp^2 t}(\overline{F}^*+X+k\int_0^t \omega(m,t')e^{kp^2 t'}dt') \quad (17)$$

Using equation (17) in equation (6) and (7), one obtains the radial and axial displacement U and W as

$$U=\frac{a_t}{4}\frac{(1+\nu)}{(1-\nu)}\sum_{m,n=1}^{\infty}\frac{P_n(z)}{\mu_m\lambda_n}e^{-kp^2 t}(\overline{F}^*+X+k\int_0^t \omega(m,t')e^{kp^2 t'}dt') \times[r^2 S_0'(k_1,k_2,\mu_m r)+2rS_0(k_1,k_2,\mu_m r)] \quad (18)$$

$$W=\frac{a_t r^2}{4}\frac{(1+\nu)}{(1-\nu)}\sum_{m,n=1}^{\infty}\frac{S_0(k_1,k_2,\mu_m r)}{\mu_m\lambda_n}P_n'(z)\,e^{-kp^2 t}(\overline{F}^*+X+k\int_0^t \omega(m,t')e^{kp^2 t'}dt') \quad (19)$$

Using equations (18) and (19) in equations (10) to (13), the stress functions are obtained as

$$\sigma_r=\frac{a_t}{4}\frac{(1+\nu)}{(1-\nu)}\sum_{m,n=1}^{\infty}\frac{e^{-kp^2 t}}{\mu_m\lambda_n}(\overline{F}^*+X+k\int_0^t \omega(m,t')e^{kp^2 t'}dt') \times\{(\lambda+2G)P_n(z)\,[r^2 S_0{}'' k_1,k_2,\mu_m r)+4rS_0'(k_1,k_2,\mu_m r) +2S_0(k_1,k_2,\mu_m r)]+\lambda[P_n(z)(rS_0'(k_1,k_2,\mu_m r)+2S_0(k_1,k_2,\mu_m r)]+ r^2 P_n''(z)S_0(k_1,k_2,\mu_m r)]\} \quad (20)$$

$$\sigma_z=\frac{a_t}{4}\frac{(1+\nu)}{(1-\nu)}\sum_{m,n=1}^{\infty}\frac{e^{-kp^2 t}}{\mu_m\lambda_n}(\overline{F}^*+X+k\int_0^t \omega(m,t')e^{kp^2 t'}dt') \times\{(\lambda+2G)r^2 P_n''(z)S_0(k_1,k_2,\mu_m r)+\lambda P_n(z)[r^2 S_0'''(k_1,k_2,\mu_m r) +5rS_0'(k_1,k_2,\mu_m r)+4S_0(k_1,k_2,\mu_m r)]\} \quad (21)$$

$$\sigma_\theta=\frac{a_t}{4}\frac{(1+\nu)}{(1-\nu)}\sum_{m,n=1}^{\infty}\frac{e^{-kp^2 t}}{\mu_m\lambda_n}(\overline{F}^*+X+k\int_0^t \omega(m,t')e^{kp^2 t'}dt') \times\{(\lambda+2G)P_n(z)(rS_0'(k_1,k_2,\mu_m r)+2S_0(k_1,k_2,\mu_m r) +\lambda[P_n(z)(r^2 S_0''(k_1,k_2,\mu_m r)+4rS_0'(k_1,k_2,\mu_m r) +2S_0(k_1,k_2,\mu_m r))+P_n''(z)S_0(k_1,k_2,\mu_m r)]\} \quad (22)$$

$$\tau_{rz}=\frac{Ga_t}{2}\frac{(1+\nu)}{(1-\nu)}\sum_{m,n=1}^{\infty}\frac{P_n'(z)}{\mu_m\lambda_n}e^{-kp^2 t}\,(\overline{F}^*+X+k\int_0^t \omega(m,t')e^{kp^2 t'}dt') \times(r^2 S_0'(k_1,k_2,\mu_m r)+2rS_0(k_1,k_2,\mu_m r)] \quad (23)$$

APPLICATIONS

Set $F(r,z,t)=\delta(r)(z-h)^2(z+h)^2 e^t$, $r=0.75$ (24)

Applying finite Marchi Fasulo transform and Marchi-Zgrablich transform to equation (30), one obtains

$$\overline{F}^*=3(k_3+k_4)e^t S_0(k_1,k_2,0.75\mu_n) \times\frac{a_n h\cos^2(a_n h)-\cos(a_n h)\sin(a_n h)}{\lambda_n a_n^2} \quad (25)$$

Substituting equation (31) in equation (22), we obtain

$$T = \sum_{m,n=1}^{\infty} \frac{S_0(k_1,k_2,\mu_m r)P_n(z)}{\mu_m \lambda_n} e^{-kp^2 t}[3(k_3+k_4)e^t S_0(k_1,k_2,0.75\mu_n) \times \frac{a_n h\cos^2(a_n h)-\cos(a_n h)\sin(a_n h)}{\lambda_n a_n^2} + X + k\int_0^t w(m,t')e^{kp^2 t'}dt'] \quad (26)$$

GRAPHICAL ANALYSIS

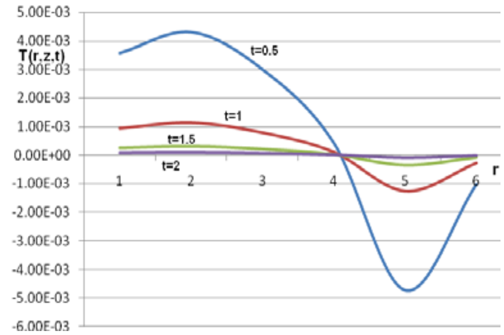

Fig. 1: T(r, z, t) versus r for different values of t

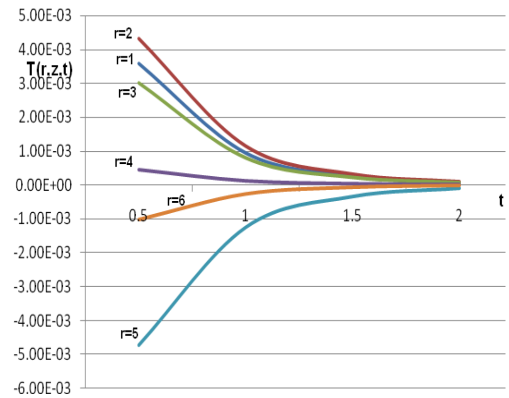

Fig. 2: T(r, z, t) versus t for different values of r

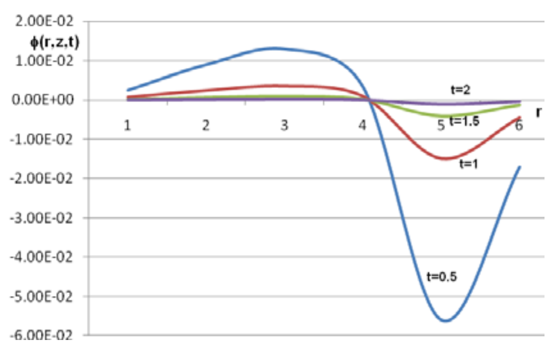

Fig. 3: ϕ (r, z, t) versus r for different values of t

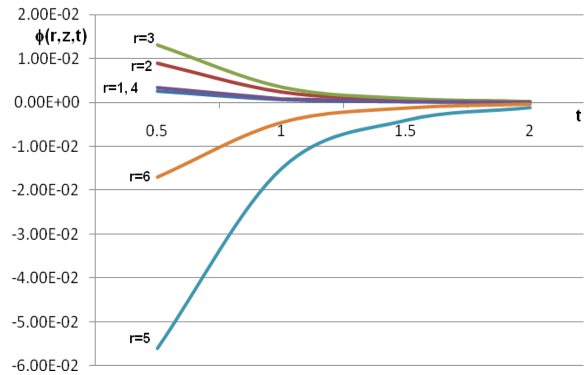

Fig. 4: ϕ (r, z, t) versus t for different values of r

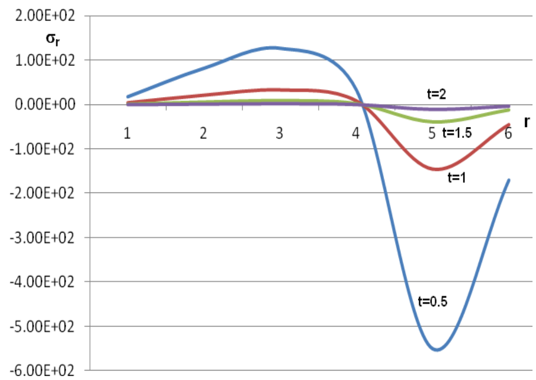

Fig. 5: σ_r versus r for different values of t

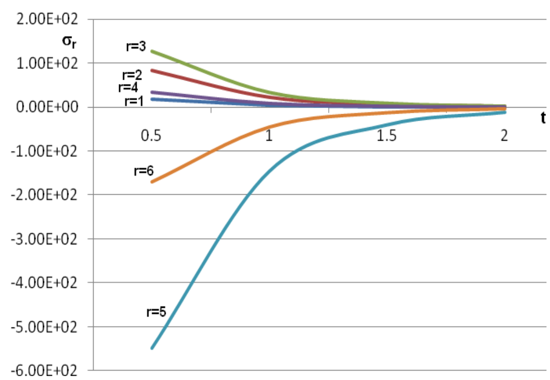

Fig. 6: σ_r versus t for different values of r

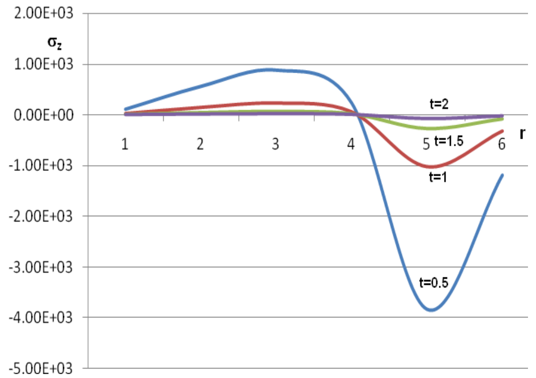

Fig. 7: σ_z versus r for different values of t

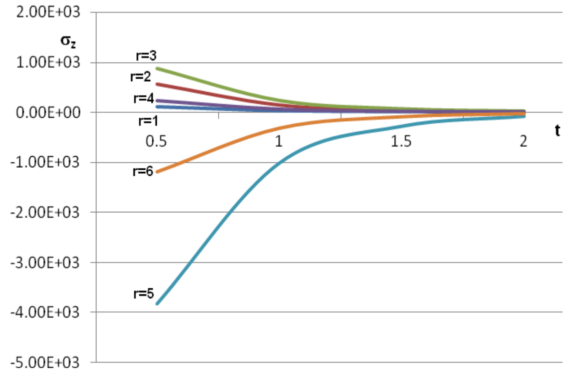

Fig. 8: σ_z versus t for different values of r

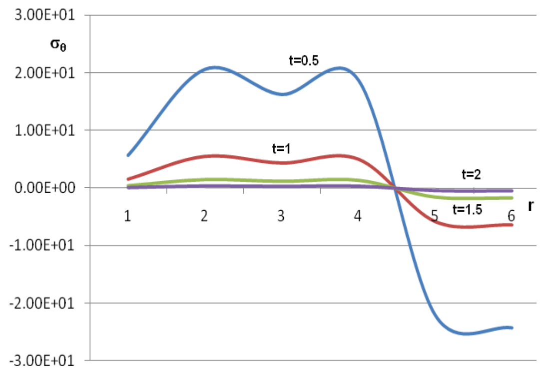

Fig. 9: σ_θ versus r for different values of t

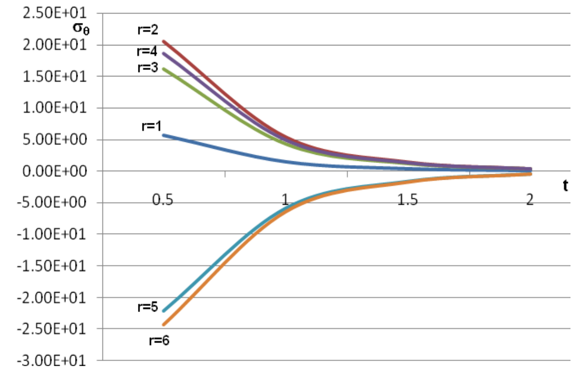

Fig. 10: σ_θ versus t for different values of r

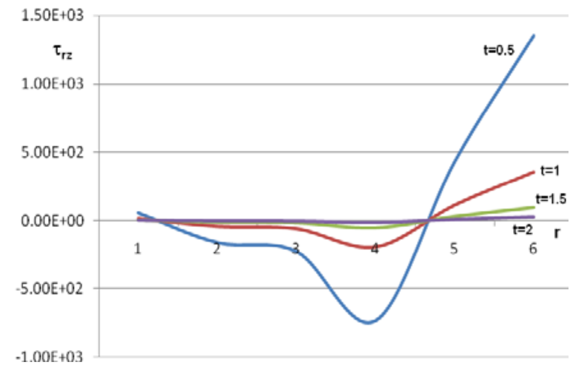

Fig. 11: τ_{rz} versus r for different values of t

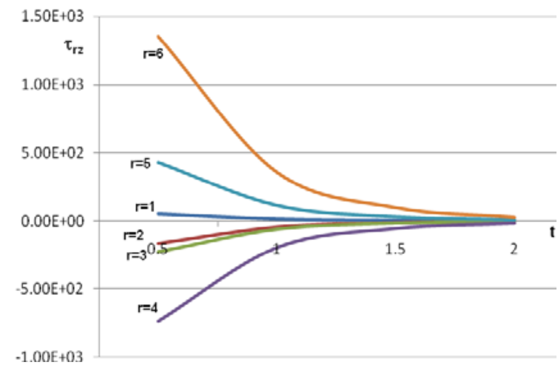

Fig. 12: τ_{rz} versus t for different values of r

CONCLUSION

In this paper, we discussed completely the inverse unsteady-state problem of thermoelastic deformation of a finite length hollow cylinder for upper plane surface where the temperature is maintained at zero on the curved surface and the lower plane surface of the cylinder respectively. The integral transform techniques are used to obtain the numerical results. The temperature, displacement and thermal stresses that are obtained can be applied to the design of useful structures and machines in engineering applications.

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Appendix B

The finite Marchi-Fasulo integral transform of f(z), -h < z < h is defined to be

$$\overline{F}(n) = \int_{-h}^{h} f(z) P_n(z) dz \qquad (1)$$

then at each point of (-h, h) at which f(z) is continuous,

$$f(z) = \sum_{n=1}^{\infty} \frac{\overline{F}(n)}{\lambda_n} P_n(z) \qquad (2)$$

where

$$P_n(z) = Q_n \cos(a_n z) - W_n \sin(a_n z)$$

$$Q_n = a_n(\alpha_1 + \alpha_2)\cos(a_n h) + (\beta_1 - \beta_2)\sin(a_n h)$$

$$W_n = (\beta_1 + \beta_2)\cos(a_n h) + (\alpha_2 - \alpha_1) a_n \sin(a_n h)$$

$$\lambda_n = \int_{-h}^{h} P_n^{\,2}(z)\, dz = h\left[Q_n^{\,2} + W_n^{\,2}\right] + \frac{\sin(2a_n h)}{2a_n}\left[Q_n^{\,2} - W_n^{\,2}\right]$$

The Eigen values a_n are the solutions of the equation

$$\left[\alpha_1 a \cos(ah) + \beta_1 \sin(ah)\right] \times \left[\beta_2 \cos(ah) + \alpha_2 a \sin(ah)\right]$$
$$= \left[\alpha_2 a \cos(ah) - \beta_2 \sin(ah)\right] \times \left[\beta_1 \cos(ah) - \alpha_1 a \sin(ah)\right] \qquad (3)$$

α_1, α_2, β_1 and β_2 are constants.

The sum in (2) must be taken on n corresponding to the positive roots of the equation (3)

Moreover the integral transform (1) has the following property:

$$\int_{-h}^{h} \frac{\partial^2 f(z)}{\partial z^2} P_n(z) dz = \frac{P_n(h)}{\alpha_1}\left[\beta_1 f(z) + \alpha_1 \frac{\partial f(z)}{\partial z}\right]_{z=h} - \frac{P_n(-h)}{\alpha_2}\left[\beta_2 f(z) + \alpha_2 \frac{\partial f(z)}{\partial z}\right]_{z=-h}$$
$$- a_n^{\,2}\overline{F}(n)$$