ON THE NON-HOMOGENEOUS TERNARY CUBIC EQUATION $(x+y)^2 - 3xy = 12z^3$

M.A.Gopalan[1], Sharadha Kumar[2]
[1]Professor, Department of Mathematics, SIGC, Trichy, Tamilnadu, India.
[2]M.Phil Scholar, Dept. of Mathematics, SIGC, Trichy, Tamilnadu, India.

Abstract

The non-homogeneous cubic equation with three unknowns represented by $(x + y)^2 - 3xy = 12z^3$ is analysed for its patterns of non-zero distinct integer solutions. A few interesting relations among the solutions are presented.

Keywords: Non-homogeneous cubic, ternary cubic, integer solutions.

Notations:

1. $t_{m,n} = n\left(1+\frac{(n-1)(m-2)}{2}\right)$ -Polygonal number of rank n with size m
2. $CP_{3,n} = \frac{n^3+n}{2}$ - Centered triangular pyramidal number of rank n
3. $CP_{6,n} = n^3$ - Centered hexagonal pyramidal number of rank n
4. $P_n^5 = \frac{n^2(n+1)}{2}$ - Pentagonal pyramidal number of rank n
5. $PR_n = n(n+1)$ - Pronic number bof rank n

INTRODUCTION

It is well known that the Diophantine equations are rich in variety [1-3]. In particular, one may refer [4-12] for cubic with three unknowns. In this paper yet another cubic equation with three unknowns is given by $(x + y)^2 - 3xy = 12z^3$ is considered for determining its infinitely many non-zero integer solutions. Also, A few interesting relations among the solutions are exhibited.

Method of analysis:

The non-homogeneous cubic equation to be solved is

$$(x+y)^2 - 3xy = 12z^3 \quad (1)$$

Introducing the linear transformations

$$x = u+v \ , \ y = u-v \quad (2)$$

in (1), it gives

$$u^2 + 3v^2 = 12z^3 \quad (3)$$

Assume

$$z = z(a,b) = a^2 + 3b^2 \quad (4)$$

Also, $12 = (3+i\sqrt{3})(3-i\sqrt{3})$ (5)

Substituting (4), (5) in (3) and applying the method of factorization, we have

$$(u+i\sqrt{3}v)(u-i\sqrt{3}v) = (3+i\sqrt{3})(3-i\sqrt{3})(a+i\sqrt{3}b)^3(a-i\sqrt{3}b)^3$$

Equating the positive and negative terms in the above equation, we have

$$(u+i\sqrt{3}v) = (3+i\sqrt{3})(a+i\sqrt{3}b)^3 \quad (6)$$

$$(u-i\sqrt{3}v) = (3-i\sqrt{3})(a-i\sqrt{3}b)^3 \quad (7)$$

Equating the real and imaginary parts in either (6) or (7), we have

$$u = 3a^3 - 27ab^2 - 9a^2b + 9b^3$$

$$v = 9a^2b - 9b^3 + a^3 - 9ab^2$$

Substituting the above values of u and v in (2), we get

$$x = x(a,b) = 4a^3 - 3ab^2 \quad (8)$$

$$y = y(a,b) = 2a^3 - 18ab^2 - 18a^2b + 18b^3 \quad (9)$$

Thus, (4), (8) and (9) represent the integer solutions to (1)

Properties:

1. $z(a,a+1) - t_{10,a} \equiv 0 \pmod 3$
2. $6[4 - x(1,b)]$ is a nasty number.

3. $x(1,b)+z(1,b)+33t_{4,b}=5$
4. $y(a,1)-2CP_{6,a}-18PR_a=18$

Note: It is worth to mention that , in addition to (5) , 12 may also be written in the following ways :

way 1: $12=(-3+i\sqrt{3})(-3-i\sqrt{3})$ (10)

way 2: $12=(i2\sqrt{3})(-i2\sqrt{3})$ (11)

Following the procedure as above , the corresponding integer solutions to (10) and (11) are presented below:

Solution for (10):

$x=-2a^3+18ab^2+18b^3-18a^2b$

$y=-4a^3+36ab^2$

$z=a^2+3b^2$

Solution for (11):

$x=18b^3-18a^2b+2a^3-18ab^2$

$y=18b^3-18a^2b-2a^3+18ab^2$

$z=a^2+3b^2$

However, we have other choices of integer solutions to (1) that are illustrated below:

$u^2+3v^2=12z^3\times 1$ (12)

Assume $1=\frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4}$ (13)

Substituting (4), (5) and (13) in (12) and employing the method of factorization, define

$(u+i\sqrt{3}v)=\frac{1}{2}(3+i\sqrt{3})(1+i\sqrt{3})(a+i\sqrt{3}b)^3$ (14)

Equating the real and imaginary parts in (14), we have

$u=-18a^2b+18b^3$

$v=2a^3-18ab^2$

Substituting the above values of u and v in (2), we get

$x=x(a,b)=-18a^2b+18b^3+2a^3-18ab^2$ (15)

$y=y(a,b)=-18a^2b+18b^3-2a^3+18ab^2$ (16)

Thus, (4), (15) and (16) represent the integer solution to (1)

Properties:

1. $z(b+1,b)-t_{10,a}\equiv 1(\mathrm{mod}\,5)$
2. $x(1,b)-18CP_{6,b}+18PR_b=2$
3. $6z(a,a^2)$ is a nasty number when $a=\frac{1}{2\sqrt{3}}\left[(2+\sqrt{3})^{n+1}-(2-\sqrt{3})^{n+1}\right], n=-1,0,1,....$
4. $y(a,1)-x(a,1)+8CP_{3,a}\equiv 0(\mathrm{mod}\,2)$
5. $6\left[y(a,1)-x(a,1)+8P_a^5+81\right]$ is a nasty number.

Note: It is to be noted that, in addition to (13) , 1 may also be represented as

$1=\frac{(1+i4\sqrt{3})(1-i4\sqrt{3})}{49}$ (17)

For this choice, the corresponding integer solutions to (1) are given by

$x=x(A,B)=49\left[4A^3+144B^3-36AB^2-144A^2B\right]$

$y=y(A,B)=49\left[-22A^3+90B^3+198AB^2-90A^2B\right]$

$z=z(A,B)=49\left[A^2+3B^2\right]$

Further, considering (10) with (13) , the integer solution to (1) are given by

$x=-4a^3+36ab^2$

$y=-2a^3+18ab^2+18a^2b-18b^3$

$z=a^2+3b^2$

Considering (10) with (17) , the integer solution to (1) are found to be

$x=x(A,B)=49\left[-90A^2B+198AB^2-22A^3-54B^3\right]$

$y=y(A,B)=49\left[54A^2B+234AB^2-26A^3-54B^3\right]$

$z=z(A,B)=49\left[A^2+3B^2\right]$

Considering (11) with (13), the integer solutions are represented as

$x=-2a^3+18b^3+18ab^2-18a^2b$

$y=-4a^3+36ab^2$

$z=a^2+3b^2$

Considering (11) with (17) the integer solutions are obtained as

$x=x(A,B)=49\left[54A^2B+234AB^2-26A^3-54B^3\right]$

$y=y(A,B)=49\left[144A^2B+36AB^2-4A^3-144B^3\right]$

$z=z(A,B)=49\left[A^2+3B^2\right]$

REFERENCES

[1] Dickson. L.E., "History of the Theory of Numbers", Vol 2, Diophantine analysis, New York, Dover, 2005.

[2] Mordell. L.J., "Diophantine Equations", Academic Press, New York, 1969.

[3] Carmichael. R.D., "The Theory of numbers and Diophantine Analysis" New York, 1959.

[4] Gopalan. M.A., Geetha. K., On the Ternary cubic Diophantine equation $x^2+y^2-xy=12z^3$", Bessel J Math., 3(2), 2013, 119-123.

[5] Gopalan.M.A., Vidhyalakshmi.S., Kavitha.A., Observation on the Ternary cubic equation $x^2 + y^2 + xy = 12z^3$, Antarctia J Math., 10(5), 2013,453-460.
[6] Gopalan.M.A., Vidhyalakshmi.S., Lakshmi.K., Lattice points on the non-homogeneous cubic equation $x^3 + y^3 + z^3 + (x + y + z) = 0$, Impact J sci Tech., vol 7,No.1,2013,21-25.
[7]Gopalan.M.A.,Vidhyalakshmi.s.,Lakshmi.K., Lattice points on the non-homogeneous cubic equation $x^3 + y^3 + z^3 - (x + y + z) = 0$, Impact J sci Tech., vol 7,No.1,2013,51-55.
[8] Gopalan.M.A., Vidhyalakshmi.S., and Mallika.S., On the Ternary non-homogeneous cubic equation $x^3 + y^3 - 3(x + y) = 2(3k^2 - 2)z^3$, Impact J sci Tech., vol 7,No.1,2013,41-45.
[9] K.Meena, M.A.Gopalan,S.Vidhyalakshmi and S.Arthy Thangam, On the ternary non-homogeneous cubic equation $4(x + y)^2 - 7xy + (x + y) + 15(x - y) = 16(z^3 - 1)$, Bassel J Math.,4(3) (2014),75-80.
[10] S.Vidhyalakshmi, M.A.Gopalan and A.Kavitha , On the ternary cubic equation $5(X + Y)^2 - 7XY + X + Y + 1 = 23Z^3$, IJIRR, 1(10) (2014) 99-101.
[11] M.A.Gopalan, S.Vidhyalakshmi and N.Thiruniraiselvi, On the non-homogeneous ternary cubic Equation $2a^2(x^2 + y^2) - 2a(k + 1)(x + y) + (k + 1)^2 = 2^{2n} z^3$, Universe of Emerging Technologies and science, 2(1)(2015)1-5.
[12] M.A.Gopalan, N.Thiruniraiselvi, A.Kavitha, On the ternary cubic Diophantine equation $7x^2 - 4y^2 = 3z^3$, International Journal of Recent Scientific Research , 6(9) (2015) 6197-6199.