TOTAL UNIDOMINATING FUNCTIONS AND TOTAL UNIDOMINATION NUMBER OF A 3-REGULARIZED WHEEL

V.Anantha Lakshmi[1], B.Maheswari[2]

[1]Lecturer, Department of Mathematics, P.R.Govt. College (A), Kakinada, Andhra Pradesh, India,

[2]Professor, Department of Applied Mathematics, S.P.Mahila Visvavidyalayam, Tirupati, Andhra Pradesh, India

Abstract

The theory of Domination in graphs is a rapidly growing area of research in Graph Theory. Domination in graphs has applications to several fields such as school bus routing, computer communication networks, Facility location problems, locating radar stations problem etc. Recently dominating functions in domination theory have received much attention. The concept of total unidominating function was introduced in [6]. The total unidominating functions of a cycle were studied in [7]. In this paper the authors define a graph named as 3-regularised wheel and study the total unidominating functions of this graph and determined its total unidomination number and the number of total unidominating functions with minimum weight.

Key words: Wheel, 3-regularized wheel, total unidominating function, total unidomination number.

1. INTRODUCTION

Graph theory has in numerous applications in different areas such as Physical Sciences, Biological Sciences and other branches of Mathematics etc. In addition, graph theory plays an important role in several areas of computer science such as switching theory, logical design etc.

Theory of domination is an important branch of graph theory that has applications in to several fields such as School bus routing, Computer communication networks, Facility location problems, Locating radar stations problem etc. Domination and its properties have been extensively studied by T.W.Haynes et.al [1, 2].

Recently dominating functions in domination theory have received much attention. Hedetniemi et.al.[3] introduced the concept of dominating function. The concept of total dominating functions was introduced by Cockayne et al. [4]. Some inequalities relating to domination parameters in cubic graphs were studied in [5]. The concept of total unidominating function is introduced and studied the total unidominating functions of a path in [6], total unidominating functions of a cycle in [7].

In this paper we define a graph named as 3-regularised wheel and find the total unidomination number of a 3-regularised wheel, the number of total unidominating functions with minimum weight. Further the results obtained are illustrated.

3- Regularized wheel is defined as "A graph formed from $W_{1,n}$ by replacing the center of $W_{1,n}$ by a cycle C_n and each of the remaining n vertices in $W_{1,n}$ are replaced by cycles C_3 ".

2. TOTAL UNIDOMINATING FUNCTIONS AND TOTAL UNIDOMINATION NUMBER

In this section the concepts of total unidominating function and total unidomination number are introduced and defined as follows:

Definition 2.1: Let $G(V,E)$ be a connected graph. A function $f: V \to \{0,1\}$ is said to be a **total unidominating function,** if

$$\sum_{u \in N(v)} f(u) \geq 1 \ \forall \ v \in V \ and \ f(v) = 1,$$

$$\sum_{u\in N(v)} f(u) = 1 \ \forall\, v \in V \ \ and \ \ f(v) = 0,$$

where $N(v)$ is the open neighbourhood of the vertex v.

Definition 2.2: The **total unidomination number** of a connected graph $G(V,E)$ is defined as

$\min\{f(V)/f$ is a total unidominating function$\}$.

It is denoted by $\gamma_{tu}(G)$.

Here $f(V) = \sum_{u\in V} f(u)$ is called as the weight of the total unidominating function f.

3. TOTAL UNIDOMINATION NUMBER OF A 3- REGULARIZED WHEEL

Theorem 3.1: The total unidomination number of a 3-regularized wheel is $\gamma_u(C_n) + n$.

Proof: Let $W_{1,n}$ be a wheel and C_n be the cycle replacing the center of $W_{1,n}$ and $C_3^1, C_3^2, \dots, C_3^n$ are the cycles replacing the n vertices in $W_{1,n}$ respectively.

Let $u_1, u_2, \dots\dots, u_n$ be the vertices in C_n, and $v_1, v_2, \dots\dots, v_n$ be the vertices in $C_3^1, C_3^2, \dots, C_3^n$ respectively which are adjacent to $u_1, u_2, \dots\dots, u_n$ respectively. Let $w_1, w_2; w_3, w_4; \dots; w_{2n-1}, w_{2n}$ be the remaining vertices in $C_3^1 C_3^2, \dots, C_3^n$ respectively.

Here $d(u_i) = d(v_i) = d(w_{2i}) = d(w_{2i-1}) = 3$ for $i = 1,2,\dots,n$.

Let g be a unidominating function of C_n with minimum weight $\gamma_u(C_n)$. where $\gamma_u(C_n)$ is the unidomination number of the cycle C_n obtained in [8].

Define a function $f: V \to \{0,1\}$ by

$$f(v) = \begin{cases} g(v) & when\ v = u_i, \quad i = 1,2,\dots,n, \\ 1 & when\ v = v_i \ and\ g(u_i) = 1, \\ 1 & when\ v = w_{2i}, w_{2i+1} \ and\ g(u_i) = g(u_{i+1}) = 0, \\ 0 & otherwise. \end{cases}$$

Now we prove that f is a total unidominating function.

Case 1: Let $g(u_i) = 1$ for some $i = 1,2,\dots,n$.Then it follows that

$f(u_i) = 1, f(v_i) = 1, f(w_{2i}) = 0, f(w_{2i+1}) = 0.$

Then $$\sum_{u\in N(u_i)} f(u) = f(u_{i-1}) + f(u_{i+1}) + f(v_i) \geq f(v_i) = 1.$$

$$\sum_{u\in N(v_i)} f(u) = f(u_i) + f(w_{2i-1}) + f(w_{2i}) = 1 + 0 + 0 = 1.$$

$$\sum_{u\in N(w_{2i-1})} f(u) = f(\mathrm{v}_i) + f(\mathrm{w}_{2i-2}) + f(\mathrm{w}_{2i}) = 1 + 0 + 0 = 1.$$

$$\sum_{u\in N(w_{2i})} f(u) = f(v_i) + f(w_{2i-1}) + f(w_{2i+1}) = 1 + 0 + 0 = 1.$$

Case 2: Let $g(u_i) = 0$ and $g(u_{i+1}) = 0$ for some i $= 1,2,\dots,n$. Then it follows that

$f(u_i) = 0, f(v_i) = 0, f(w_{2i-1}) = 0, f(w_{2i}) = 1, f(w_{2i+1}) = 1, f(w_{2i+2}) = 0.$

Then $$\sum_{u\in N(u_i)} f(u) = f(u_{i-1}) + f(u_{i+1}) + f(v_i) = 1 + 0 + 0 = 1.$$

$$\sum_{u\in N(v_i)} f(u) = f(\mathrm{u}_i) + f(\mathrm{w}_{2i-1}) + f(\mathrm{w}_{2i}) = 0 + 0 + 1 = 1,$$

$$\sum_{u\in N(v_{i+1})} f(u) = f(\mathrm{u}_{i+1}) + f(\mathrm{w}_{2i+1}) + f(\mathrm{w}_{2i+2}) = 0 + 1 + 0 = 1,$$

$$\sum_{u\in N(w_{2i+1})} f(u) = f(w_{2i}) + f(v_{i+1}) + f(w_{2i+2}) = 1 + 0 + 0 = 1,$$

$$\sum_{u\in N(w_{2i+2})} f(u) = f(w_{2i+1}) + f(v_{i+1}) + f(w_{2i+3}) = 1 + 0 + 0 = 1.$$

From Case 1 and Case 2 it follows that f is a total unidominating function.

From the definition of f, we have

$$\sum_{i=1}^{n} f(u_i) = \sum_{i=1}^{n} g(u_i) = \gamma_u(C_n), \sum_{i=1}^{n} f(v_i) = \sum_{i=1}^{n} g(u_i) = \gamma_u(C_n),$$

$$\sum_{i=1}^{n} f(w_{2i-1}) + \sum_{i=1}^{n} f(w_{2i}) = \frac{1}{2}\left(n - \sum_{i=1}^{n} g(u_i) + n - \sum_{i=1}^{n} g(u_i)\right)$$

$$= \frac{1}{2}[2n - 2\gamma_u(C_n)] = n - \gamma_u(C_n).$$

Therefore $$\sum_{u\in V} f(u) = \sum_{i=1}^{n} f(u_i) + \sum_{i=1}^{n} f(v_i) + \sum_{i=1}^{n} f(w_{2i-1}) + \sum_{i=1}^{n} f(w_{2i})$$

$$= \gamma_u(C_n) + \gamma_u(C_n) + n - \gamma_u(C_n) = \gamma_u(C_n) + n.$$

By the definition of total unidomination number, it follows that

$$\gamma_{tu}(3 - regularised\ wheel) \leq \gamma_u(C_n) + n - - - (1)$$

Let f be a total unidominating function.
Then f has the following properties.

1. If $f(u_i) = 0$ and $f(u_{i-1}) = 1$ *or* $f(u_{i+1}) = 1$ then $f(v_i)$ must be 0 and $f(w_{2i}) = 1$ *or* $f(w_{2i+1}) = 1$ respectively. Otherwise if $f(u_i) = 0$ and both of $f(u_{i-1}), f(u_{i+1})$ are 0 then $f(v_i), f(w_{2i-1})$, $f(w_{2i})$ must be 1.
2. If $f(u_i) = 1$ and both of $f(u_{i-1}), f(u_{i+1})$ are 0 then $f(v_i)$ must be 1.

Let k_1 be the number of $u_i s$ such that $f(u_i) = 0$ and any one of $f(u_{i-1}), f(u_{i+1})$ is 1 then $0 \le k_1 \le n - \gamma_u(C_n)$ and $\sum_{k_1} f(v_i) + f(w_{2i-1}) + f(w_{2i}) = k_1$ for these k_1 sets of vertices (v_i, w_{2i-1}, w_{2i}) where i is such that $f(u_i) = 0$ $\mathit{and}\ f(u_{i-1}) = 1 \quad \mathit{or}\ f(u_{i+1}) = 1$.

Let k_2 be the number of $u_i s$ such that $f(u_i) = 0$ and $f(u_{i-1}) = f(u_{i+1}) = 0$ then $0 \le k_2 \le n$ and $\sum_{k_2} f(v_i) + f(w_{2i-1}) + f(w_{2i}) = 3k_2$ for these k_2 sets of vertices (v_i, w_{2i-1}, w_{2i}), where i is such that $f(u_i) = 0\ \mathit{and}\ f(u_{i-1}) = f(u_{i+1}) = 0$.

Then there are $n - (k_1 + k_2) u_i s$ such that $f(u_i) = 1$ and $\sum f(v_i) + f(w_{2i-1}) + f(w_{2i}) \ge n - (k_1 + k_2)$ for these $n - (k_1 + k_2)$ sets of vertices (v_i, w_{2i-1}, w_{2i}) where i is such that $f(u_i) = 1$.

Therefore $f(V)$

$$= \sum_{i=1}^{n} f(u_i) + \sum_{k_1} f(v_i) + f(w_{2i-1}) + f(w_{2i})$$

$$+ \sum_{k_2} f(v_i) + f(w_{2i-1}) + f(w_{2i})$$

$$+ \sum_{n-(k_1+k_2)} f(v_i) + f(w_{2i-1}) + f(w_{2i})$$

$$\ge n - (k_1 + k_2) + k_1 + 3k_2 + n - (k_1 + k_2)$$

$$= 2n - k_1 + k_2$$

$$\ge 2n - \left(n - \gamma_u(C_n)\right) + k_2 \ge n - \gamma_u(C_n).$$

Since f is defined arbitrarily, it follows that

$$\gamma_{tu}(3 - \mathit{regularised\ wheel}) \ge \gamma_u(C_n) + n --- (2)$$

Therefore from the inequalities (1) and (2), we get

$$\gamma_{tu}(3 - \mathit{regularized\ \ wheel}) = \gamma_u(C_n) + n.$$

Theorem 3.2: The number of total unidominating functions of a 3-regularized wheel with minimum weight $\gamma_u(C_n) + n$ is the number of unidominating functions of C_n with minimum weigh $\gamma_u(C_n)$.

Proof: Consider the total unidominating function f with minimum weight $\gamma_u(C_n) + n$ given in Theorem 3.1. As the function f is given in terms of g, a unidominating function of C_n, it is clear that the number of total unidominating functions of 3 - regularized graph with minimum weight is equal to the number of unidominating functions of C_n with minimum weight.

Therefore the number of total unidominating functions of a 3- regularized wheel are

$$\begin{cases} 3 & \mathit{when} \quad n \equiv 0(\mathit{mod}\ 3), \\ n & \mathit{when}\ \ n \equiv 1(\mathit{mod}\ 3), \\ n\left(1 + \left\lfloor \frac{n}{6} \right\rfloor\right) & \mathit{when} \quad n \equiv 2(\mathit{mod}\ 3), n \neq 8, \\ 12 & \mathit{when} \quad n = 8. \end{cases}$$

Now we verify that whether there is any other total unidominating function with minimum weight.

Let f be a total unidominating function. Then we have proved in Theorem 3..1 that

$$f(V) \ge\ 2n - k_1 + k_2.$$

If $k_1 = k_2 = 0$then$f(V) \ge\ 2n > \gamma_u(C_n) + n$.

If $k_1 = 0, k_2 > 0$ then $f(V) >\ 2n > \gamma_u(C_n) + n$.

If $k_1 < n - \gamma_u(C_n), k_2 > 0$ then $f(V) \ge\ 2n - k_1 + k_2 > 2n - n + \gamma_u(C_n) = \gamma_u(C_n) + n$.

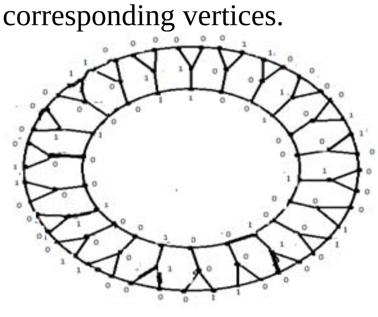
In the above three cases we have $f(V) > \gamma_u(C_n) + n$. Therefore f is not a function with minimum weight.

If $k_1 = n - \gamma_u(C_n), k_2 = 0$ then $f(V) = \gamma_u(C_n) + n$ and this function coincides with one of the above said functions.

Therefore there is no other total unidominating function with minimum weight.

4 ILLUSTRATIONS

Example 4.1: The functional values of a total unidominating functions of a 3- regularized wheel formed from $W_{1,22}$ are given at the corresponding vertices.



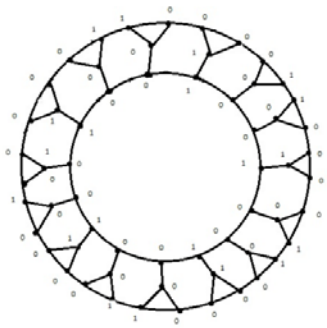
3- regularized wheel formed from $\boldsymbol{W_{1,22}}$

Total unidomination number of 3- regularized wheel formed from $W_{1,22}$ is

$$\gamma_{tu}(3-\text{regularised wheel}) = \gamma_u(C_{22}) + 22$$
$$= \left\lceil\frac{22}{3}\right\rceil + 22 = 8 + 22 = 30.$$

There are 22 total unidominating functions of the 3-regularized wheel formed from $W_{1,22}$ having the minimum weight 30. ■

Example 4.2: The functional values of a total unidominating functions of a 3- regularised wheel formed from $W_{1,15}$ are denoted at the corresponding vertices.



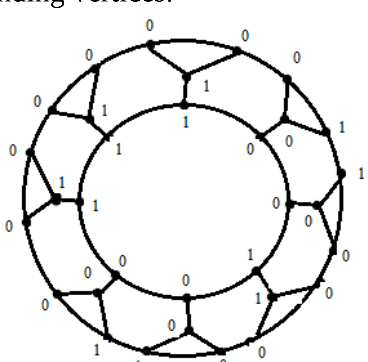
3-Regular wheel formed from $\boldsymbol{W_{1,15}}$

Total unidomination number of the regular wheel formed from $W_{1,15}$is

$$\gamma_{tu}(3-regular\ wheel) = \gamma_u(C_{15}) + 15$$
$$= 20.$$

Number of total unidominating functions of the 3-regularized wheel formed from $W_{1,15}$ having the minimum weight 20 are 3. ■

Example 4.3: The functional values of a total unidominating functions of a 3- regularized wheel formed from $W_{1,8}$ are denoted at the corresponding vertices.



3-Regular wheel formed from $\boldsymbol{W_{1,8}}$

Total unidomination number of the regular wheel formed from $W_{1,8}$is

$$\gamma_{tu}(3-regular\ wheel\ formed\ from\ W_{1,8})$$
$$= \gamma_u(C_8) + 8 = 12.$$

Number of total unidominating functions of the 3-regular wheel formed from $W_{1,8}$ having the minimum weight 12 are 12.

5.CONCLUSION

This work gives a scope to find upper total unidomination number and the number of minimal total unidominating function with maximum weight of a 3-regularized wheel.

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